

On Skein Algebras And $Sl_2(\mathbb{C})$ -Character Varieties

Józef H. Przytycki and Adam S. Sikora¹

1 Introduction

Let M be an oriented 3-manifold. For any commutative ring R with a specified invertible element A one can assign an R -module $\mathcal{S}_{2,\infty}(M; R, A)$ called the Kauffman bracket skein module of M . This invariant of 3-manifolds was introduced by the first author in [P-1].

This paper gives insight into broad and intriguing connections between two apparently unrelated theories: the theory of skein modules of 3-manifolds and the theory of representations of groups into special linear groups of 2×2 matrices. This connection was first observed by D. Bullock ([B-1],[B-2],[B-4]).

We believe that our research, which was originated in [P-S-1] and continued in this paper, will ultimately result in a theory which will reveal some of the mysterious mutual correlations between a skein approach and an approach via methods of representation theory to 3-dimensional topology.

Such a theory is needed, for example, in order to advance the study of quantum invariants. These invariants can be defined both in terms of skein theory and representation theory (of quantum groups). A lack of good understanding of relations between these two theories gives rise to some difficulties in studying quantum invariants.

Our work can be also considered in a context of the theory of Culler and Shalen ([C-S]) which relates properties of the $Sl_2(\mathbb{C})$ -character variety of $\pi_1(M)$, for a given 3-manifold M , with properties of incompressible surfaces in M . In this paper we give a topological interpretation of $Sl_2(\mathbb{C})$ -character variety of $\pi_1(M)$. Using this interpretation one can restate some of the deep results of Culler and Shalen in a purely topological manner.

We were inspired to write this paper after we discovered the results of Brumfiel and Hilden, [B-H], which interplay very nicely with our own work

¹ The second author was partially supported by NSF grant DMS93-22675.

[P-S-1]. This paper is based on these two papers. In particular, we use the following results from [P-S-1]:

1. If $A = -1$ then $\mathcal{S}_{2,\infty}(M; R, A)$ is a commutative R -algebra, called a skein algebra of M .
2. For any group G and a ring R one can define a skein algebra of G , $\mathcal{S}(G; R)$, in such a way that if $G = \pi_1(M)$ then the two notions of skein algebras coincide, i.e.

$$\mathcal{S}_{2,\infty}(M; R, -1) \simeq \mathcal{S}(G; R).$$

Skein algebras will be the main subject of our study. Here is the plan of our paper:

In Section 2 we give the definition of the Kauffman bracket skein module, $\mathcal{S}(M; R, A)$, and the skein algebra, $\mathcal{S}(G; R)$. We recall also all necessary results from [P-S-1] needed in this paper.

In Section 3 we prove that if some general conditions are satisfied (for example, if $\frac{1}{2} \in R$) then $\mathcal{S}(G; R)$ is isomorphic to an algebra $TH_R(G)$ introduced by Brumfiel and Hilden in connection with their study of $Sl_2(R)$ -representations of groups, [B-H]. Using this fact we prove a few new results about the algebras $TH_R(G)$. For example, $TH_R(\pi_1(F))$ is a free module for any ring R and for any surface F .

In Section 7 we consider characters of Sl_2 -representations of groups. We show, using the work of Brumfiel and Hilden, that if K is an algebraically closed field of characteristic 0 then the algebra $\mathcal{S}(G; K)$ is isomorphic to the (unreduced) coordinate ring of the $Sl_2(K)$ -character variety of G . Moreover, using various algebraic and topological methods, we prove that for many important classes of groups this algebra is actually reduced, i.e. the algebra $\mathcal{S}(G; K)$ does not have any nilpotent elements.

In Section 4 we consider another structure of a generally non-commutative algebra assigned to $\mathcal{S}_{2,\infty}(F \times I; R, A)$, when F is a surface. We present some of its interesting properties. For example, $\mathcal{S}_{2,\infty}(F \times I; R, A)$ is a central algebra over a ring of polynomials induced by the boundary of F . Moreover, $\mathcal{S}_{2,\infty}(F \times I; R, A)$ has no zero divisors. In particular, this implies that the $Sl_2(C)$ -character varieties of the fundamental groups of surfaces are irreducible.

In Sections 6 and 8 we find, using results of previous sections and some arguments from algebraic geometry, the minimal numbers of generators of

various skein algebras. In particular, we prove that the minimal number of generators of $\mathcal{S}(F_n, R)$ is $n + \binom{n}{2} + \binom{n}{3}$ or $2^n - 1$ depending on whether 2 is invertible in R or not. We also give the minimal numbers of generators of skein algebras $\mathcal{S}_{2,\infty}(F \times I; R, A)$, for $R = \mathbb{Z}[A^{\pm 1}]$, and estimate the minimal numbers of generators for other rings of coefficients, R .

In Section 5 we introduce for any surface F a new algebra $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$ which is built on relative links in the manifold $M = F \times [0, 1]$. We prove that this algebra is isomorphic to an algebra $H_R(G)$, $G = \pi_1(M)$, investigated in the book by Brumfiel and Hilden, [B-H]. This result, combined with our result from Section 3, gives a nice topological interpretation of algebraic objects, $H_R(G)$ and $TH_R(G)$, considered in [B-H].

2 Skein Modules and Skein Algebras

In this paper we make the following assumptions:

1. All rings are commutative and have identities. All homomorphisms between rings preserve identities.
2. For a given ring R and a set \mathcal{S} we denote the free R -module with a basis composed of elements of \mathcal{S} by $R\mathcal{S}$.
3. All topological spaces (manifolds, links) are considered as objects in the category of piecewise-linear topological spaces. In particular, all links are tame. All continuous functions preserve PL-structures.
4. All manifolds are oriented and they may have boundaries.
5. Let N be a submodule of M . Then for any $m \in M$ we denote $m + N \in M/N$ by $[m]$. Similarly, if I is an ideal in a ring R then $x + I \in R/I$ is denoted by $[x]$, for any $x \in R$.

We start with a definition of the Kauffman bracket skein module. Skein modules were independently introduced by V. Turaev in [T-1] and, in a more general setting, by the first author in [P-1]. Kauffman bracket skein modules of manifolds were for the first time defined in [P-1]. They are composed of formal linear combinations of framed unoriented links considered up to some local relations called *skein relations*. By a framed link in a 3-manifold M we mean an embedding of a finite family of annuli into the interior of M . In our definition of the Kauffman bracket skein module we follow [H-P-1].

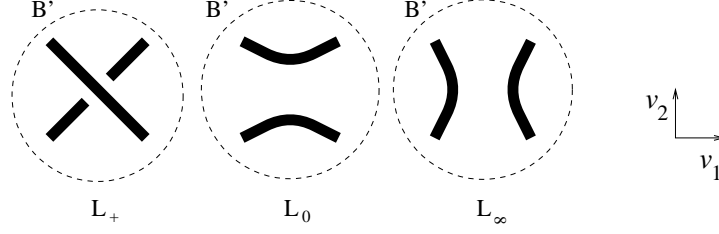


Figure 1

Definition 2.1

Let M be any oriented 3-manifold and let $\mathcal{L}_{fr}(M)$ denote the set of all ambient isotopy classes of framed unoriented links in the interior of M , including the empty link, \emptyset . Let A be a specified invertible element in a ring R . Furthermore, let $S_{2,\infty}$ be a submodule of $R\mathcal{L}_{fr}(M)$ generated by two kinds of elements:

1. skein expressions $L_+ - AL_0 - A^{-1}L_\infty$, where L_+, L_0, L_∞ are any three framed links in M which are the same outside a small 3-ball $B \subset M$, but inside B they are as in Fig. 1, i.e. there is an orientation preserving homeomorphism between B and a ball $B' \subset \mathbb{R}^3$ which carries $B \cap L_+, B \cap L_-, B \cap L_0$ to fragments of links presented in Fig. 1. The orientation of $B' \subset \mathbb{R}^3$ is given by an ordered basis of \mathbb{R}^3 , v_1, v_2, v_3 , where v_1, v_2 are pictured in Fig. 1 and v_3 is orthogonal to v_1 and v_2 and it is directed towards the reader. In this situation we say that L_+, L_0, L_∞ are skein related.
2. $L \cup \bigcirc + (A^2 + A^{-2}) \cdot L$, where L is any link in M and $L \cup \bigcirc$ denotes a disjoint union of L with a trivial component, \bigcirc . We assume that \bigcirc has a trivial framing and \bigcirc is unlinked with L i.e. \bigcirc lies in a plane in a ball disjoint from L .

We define the Kauffman bracket skein module of M as a quotient

$$\mathcal{S}_{2,\infty}(M; R, A) = R\mathcal{L}_{fr}(M)/S_{2,\infty}.$$

Notice that if a 3-manifold M is a disjoint union of manifolds M_1, M_2, \dots, M_n then

$$\mathcal{S}_{2,\infty}(M; R, A) = \mathcal{S}_{2,\infty}(M_1; R, A) \otimes_R \mathcal{S}_{2,\infty}(M_2; R, A) \otimes_R \dots \otimes_R \mathcal{S}_{2,\infty}(M_n; R, A).$$

Therefore, we will assume, for simplicity, that all manifolds are connected unless otherwise stated.

J. Barrett proved in [Bar] that the existence of a spin structure for any oriented 3-manifold M implies that for any knot K in M we can define its framing, $Spin(K) \in \mathbb{Z}_2$, in such a way that the following theorem holds

Theorem 2.2 ([Bar])

For any ring R there exists an isomorphism of R -modules $\phi : \mathcal{S}_{2,\infty}(M; R, A) \rightarrow \mathcal{S}_{2,\infty}(M; R, -A)$, such that $\phi([L]) = (-1)^{\sum Spin(K)} [L]$, where the sum is over all connected components of L .

The following fact, which is an easy generalization of results of [P-S-1], shows that skein modules $\mathcal{S}_{2,\infty}(M; R, A)$ are particularly interesting for $A = 1$ and $A = -1$.

Fact 2.3

Let M be any oriented manifold and let R be a ring.

1. *If L_1 and L_2 are two homotopic framed links or, equivalently, if L_2 can be obtained from L_1 by an ambient isotopy and a sequence of crossing changes, then $[L_1] = [L_2]$ in $\mathcal{S}_{2,\infty}(M; R, A)$ for $A = 1$ or $A = -1$.*
2. *$[L] \in \mathcal{S}_{2,\infty}(M; R, A)$, for $A = -1$, does not depend on the framing of L .*
3. *If $A = 1$ or $A = -1$ then there is a multiplication operation on $\mathcal{S}_{2,\infty}(M; R, A)$ uniquely determined by the following condition:*

For any two links L_1, L_2 in M and any disjoint sum of them, $L_1 \cup L_2$, $[L_1] \cdot [L_2] = [L_1 \cup L_2]$ in $\mathcal{S}_{2,\infty}(M; R, -1)$. This multiplication is commutative, associative and has an identity $[\emptyset]$.

We use Barrett's result to prove the following proposition.

Proposition 2.4

The isomorphism of modules, ϕ , introduced in Theorem 2.2 is an isomorphism of R -algebras $\phi : \mathcal{S}_{2,\infty}(M; R, -1) \rightarrow \mathcal{S}_{2,\infty}(M; R, 1)$.

Proof: Suppose that links L, L' have connected components K_1, K_2, \dots, K_n , and K'_1, K'_2, \dots, K'_m respectively. Then

$$\begin{aligned}\phi([L][L']) &= \phi([L \cup L']) = (-1)^{\sum_{i=1}^n \text{Spin}(K_i) + \sum_{i=1}^m \text{Spin}(K'_i)} [L \cup L'] = \\ &= (-1)^{\sum_{i=1}^n \text{Spin}(K_i)} [L] \cdot (-1)^{\sum_{i=1}^m \text{Spin}(K'_i)} [L'] = \phi([L])\phi([L']).\end{aligned}$$

Therefore ϕ is a ring homomorphism and, by Theorem 2.2, it is an isomorphism of R -algebras. \square

Although the algebras $\mathcal{S}_{2,\infty}(M; R, -1)$ and $\mathcal{S}_{2,\infty}(M; R, 1)$ are isomorphic, the skein algebra $\mathcal{S}_{2,\infty}(M; R, -1)$ is much easier to understand (for example Fact 2.3(2) is not true for $A = 1$). Therefore, whenever we focus our attention on skein algebras we will prefer the choice $A = -1$ rather than $A = 1$. We believe that understanding the properties of skein algebras $\mathcal{S}_{2,\infty}(M; R, -1)$ is the first step towards a better understanding of skein modules $\mathcal{S}_{2,\infty}(M; R, A)$, with a generic $A \in R$.

We have proved in [P-S-1] that the skein module $\mathcal{S}_{2,\infty}(M; R, -1)$ depends only on the fundamental group of the manifold M . This motivated us to generalize the notion of skein modules of manifolds to the notion of skein algebras of groups.

Definition 2.5

Let us consider a group G and a ring R . Let \mathbf{TRG} be the tensor algebra over the module RG . Let \mathcal{I} be an ideal of \mathbf{TRG} generated by $e - 2$ and expressions $g \otimes h - h \otimes g$, $g \otimes h - gh - gh^{-1}$, for any $g, h \in G$. We define the skein algebra of G as $\mathcal{S}(G; R) = \mathbf{TRG}/\mathcal{I}$.

Obviously \otimes is the multiplication in $\mathcal{S}(G; R)$ and usually $[gh] \neq [g] \otimes [h]$ in $\mathcal{S}(G; R)$. The algebra $\mathcal{S}(G; R)$ was denoted by $\mathcal{S}^-(G; R)$ in [P-S-1].

Very recently we have learned of work of K. Saito [S-1], [S-2]. He assigns for each group G (and for a given ring R) a *universal character ring* of G . This ring is isomorphic to our skein algebra $\mathcal{S}(G; R)$.

We list below a few elementary properties of $\mathcal{S}(G; R)$ which will be needed in the further part of the paper (See [P-S-1] for a proof).

Fact 2.6

1. For any $g \in G$ $[g] = [g^{-1}]$ in $\mathcal{S}(G; R)$.

2. For any $g, h \in G$ $[g] = [hgh^{-1}]$ and $[gh] = [hg]$ in $\mathcal{S}(G; R)$.
3. Any homomorphism of groups $f : G_1 \rightarrow G_2$ yields a homomorphism of R -algebras $f_* : \mathcal{S}(G_1; R) \rightarrow \mathcal{S}(G_2; R)$ such that $f_*([g]) = [f(g)]$ for any $g \in G$. Moreover, epimorphisms of groups yield epimorphisms of algebras
4. (Universal Coefficient Property). Let $r : R \rightarrow R'$ be a homomorphism of rings. We can think of R' as an R -module. Then the identity map on G induces an isomorphism of R' -algebras:

$$\bar{r} : \mathcal{S}(G; R') \rightarrow \mathcal{S}(G; R) \otimes_R R'.$$

In particular, $\mathcal{S}(G; R) \simeq \mathcal{S}(G; \mathbb{Z}) \otimes_{\mathbb{Z}} R$.

We will also use another version of the Universal Coefficient Property concerning skein modules $\mathcal{S}_{2,\infty}(M; R, A)$. The proof of it is analogous to our proof of Fact 2.6(4) given in [P-S-1] (Compare [P-1]).

Fact 2.7

Let $r : R \rightarrow R'$ be a homomorphism of rings, $A^{\pm 1} \in R$, $A' = r(A)$. Then for any 3-manifold M there exists an isomorphism of R' -modules

$$\bar{r} : \mathcal{S}_{2,\infty}(M; R', A') \rightarrow \mathcal{S}_{2,\infty}(M; R, A) \otimes_R R'$$

such that $\bar{r}([L]) = [L]$ for any framed link L in M .

We have proved the following theorem in [P-S-1] (see Lemmas 1.3 and 1.5).

Theorem 2.8

For any 3-manifold M and any ring R there exists an isomorphism

$$\hat{\xi} : \mathcal{S}_{2,\infty}(M; R, -1) \rightarrow \mathcal{S}(\pi_1(M); R),$$

such that for any knot K in M $\hat{\xi}([K]) = -[\gamma]$, where γ is an element of the fundamental group of M represented by K .

3 Group representations into $Sl_2(R)$

In this section we are going to prove that under some general conditions the skein algebra of a group G is isomorphic to the algebra $TH_R(G)$ defined and investigated in the book by Brumfiel and Hilden, [B-H].

Let G be any group and R be a ring. Let I be an ideal in the group ring RG generated by the expressions $g(h + h^{-1}) - (h + h^{-1})g$ for $g, h \in G$. We define (after Brumfiel and Hilden) an R -algebra $H_R(G)$ to be RG/I .

There exists an involution on RG carrying g to g^{-1} for any $g \in G$. Since this involution sends I to I , it yields an involution $\iota : H_R(G) \rightarrow H_R(G)$ such that $\iota([g]) = [g^{-1}]$.

Let us consider the ring $M_2(R)$ of 2×2 matrices (with coefficients in R) and an involution $\iota : M_2(R) \rightarrow M_2(R)$ defined by

$$\iota \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The following fact, observed by Brumfiel and Hilden (Proposition 1.3 [B-H]), gives a good motivation for studying the algebras $H_R(G)$.

Fact 3.1

Any homomorphism $h : G \rightarrow Sl_2(R)$ extends to a homomorphism of R -algebras $\hat{h} : H_R(G) \rightarrow M_2(R)$ preserving involutions (i.e. $\iota \circ \hat{h} = \hat{h} \circ \iota$). Any involution preserving homomorphism $\hat{h} : H_R(G) \rightarrow M_2(R)$ can be obtained in this way.

Let $TH_R(G)$ be the subalgebra of $H_R(G)$ generated by elements of the form $[g] + [g^{-1}]$, $g \in G$. One can easily notice (see [B-H]) that $TH_R(G)$ is a commutative algebra and $TH_R(G)$ is isomorphic, as an R -module, to $R + \text{Span}_R\{[g] + [g^{-1}] : g \in G\} \subset H_R(G)$.

The following theorem shows that the skein algebra of a group G is strongly related to $TH_R(G)$.

Theorem 3.2

There exists an epimorphism of R -algebras, $\psi : \mathcal{S}(G; R) \rightarrow TH_R(G)$, given by $\psi([g]) = [g] + [g^{-1}]$, for any $g \in G$. Moreover, if 2 is not a zero divisor in $\mathcal{S}(G; R)$, then ψ is an isomorphism.²

² The assumption that 2 is not a zero divisor is essential. For example, $\mathcal{S}(\mathbb{Z}_2; \mathbb{Z}_2) \simeq \mathbb{Z}_2[x]/(x^2)$ and $TH_{\mathbb{Z}_2}(\mathbb{Z}_2) \simeq \mathbb{Z}_2$.

Proof: Let $\psi_0 : RG \rightarrow TH_R(G)$ be a homomorphism of R -modules given by $\psi_0(g) = [g] + [g^{-1}]$ (ψ_0 is well defined because RG is a free R -module). We can uniquely extend ψ_0 to a homomorphism of R -algebras ψ_1 defined on the tensor algebra of RG , $\psi_1 : TRG \rightarrow TH_R(G)$. Since $TH_R(G)$ is commutative, $\psi_1(a \otimes b - b \otimes a) = 0$. Moreover,

$$\psi_1(e - 2) = [e] + [e^{-1}] - 2 = 0 \quad \text{and}$$

$$\begin{aligned} \psi_1(a \otimes b - ab - ab^{-1}) &= \psi_1(a)\psi_1(b) - \psi_1(ab) - \psi_1(ab^{-1}) = ([a] + [a^{-1}])([b] + [b^{-1}]) - \\ &= ([ab] + [b^{-1}a^{-1}]) - ([ab^{-1}] + [ba^{-1}]) = [a^{-1}][b] + [b^{-1}] - ([b] + [b^{-1}])[a^{-1}] = 0. \end{aligned}$$

Therefore ψ_1 yields a homomorphism $\psi : \mathcal{S}(G; R) \rightarrow TH_R(G)$ such that $\psi([g]) = [g] + [g^{-1}]$. Since $TH_R(G)$ is generated by the elements $[g] + [g^{-1}]$, for $g \in G$, ψ is an epimorphism.

We are going to prove that if 2 is not a zero divisor in $\mathcal{S}(G; R)$, then ψ is also a monomorphism. Before we start a proof of this fact we will introduce two lemmas.

Lemma 3.3

For any group G and any ring R the skein R -module $\mathcal{S}(G; R)$ is generated by elements $[g]$, for $g \in G$.

Proof: By definition, $\mathcal{S}(G; R)$ is generated as an R -module by elements $[g_1] \otimes [g_2] \otimes \dots \otimes [g_n]$, where $g_1, g_2, \dots, g_n \in G$. Moreover, $[g] \otimes [h] = [gh] + [gh^{-1}]$. Therefore any product $[g_1] \otimes [g_2] \otimes \dots \otimes [g_n]$ can be written as a linear combination of elements $[g_1 \cdot g_2^{\pm 1} \cdot \dots \cdot g_n^{\pm 1}]$. \square

Recall that $I \triangleleft RG$ was an ideal generated by expressions $g(h + h^{-1}) - (h + h^{-1})g$, for $g, h \in G$. We will need the following lemma.

Lemma 3.4

I is, as an R -module, generated by expressions $kg(h + h^{-1}) - k(h + h^{-1})g$, for $k, g, h \in G$.

Proof: By definition, I is spanned by the elements

$$k(g(h + h^{-1}) - (h + h^{-1})g)l,$$

for $k, g, h, l \in G$. But

$$k(g(h + h^{-1}) - (h + h^{-1})g)l = k[(g(h + h^{-1})l - gl(h + h^{-1}) + gl(h + h^{-1})$$

$$-(h + h^{-1})gl] = kg((h + h^{-1})l - l(h + h^{-1})) + k(gl(h + h^{-1}) - (h + h^{-1})gl).$$

□

Let $\phi_0 : RG \rightarrow \mathcal{S}(G; R)$ be a homomorphism of R -modules such that $\phi_0(g) = [g] \in \mathcal{S}(G; R)$, for any $g \in G$. Then, by Fact 2.6(2),

$$\phi_0(kg(h + h^{-1}) - k(h + h^{-1})g) = [kgh] + [kgh^{-1}] - [khg] - [kh^{-1}g] =$$

$$[kgh] + [kgh^{-1}] - [gkh] - [gkh^{-1}] = [kg] \otimes [h] - [gk] \otimes [h] = 0$$

for any $k, g, h \in G$. Since I is spanned by elements $kg(h + h^{-1}) - k(h + h^{-1})g$, we have $\phi_0(I) = 0$. Therefore ϕ_0 yields a homomorphism of R -modules

$$\phi : H_R(G) = RG/I \rightarrow \mathcal{S}(G; R).$$

Notice that $\phi \circ \psi([g]) = \phi([g] + [g^{-1}]) = [g] + [g^{-1}] = 2[g] \in \mathcal{S}(G; R)$. By Lemma 3.3 elements $[g], g \in G$, generate $\mathcal{S}(G; R)$. Therefore

$$\phi \circ \psi = 2 \cdot id : \mathcal{S}(G; R) \rightarrow \mathcal{S}(G; R).$$

Hence if 2 is not a zero divisor in $\mathcal{S}(G; R)$ then ψ is a monomorphism. □

Theorem 3.2 implies the following corollary.

Corollary 3.5

If one of the following conditions holds

$$(1) \frac{1}{2} \in R; \text{ or}$$

$$(2) \mathcal{S}(G; R) \text{ is a free } R\text{-module and } 2 \text{ is not a zero divisor in } R$$

then $\psi : \mathcal{S}(G; R) \rightarrow TH_R(G)$ is an isomorphism.

The condition (2) of the above corollary motivated us to consider the following question: Under what conditions on R and G the skein algebra $\mathcal{S}(G, R)$ is a free R -module ? There are only few classes of groups, G , for which we are able to prove that $\mathcal{S}(G; R)$ is free. Our proofs are based on topological methods.

Theorem 3.6

If one of the following holds

$$(1) G \text{ is an abelian group;}$$

- (2) G is a free group;
- (3) G is the fundamental group of a surface;
- (4) G is the knot group of a $(2, 2k + 1)$ -torus knot, $k \geq 0$;

then the skein algebra $\mathcal{S}(G; R)$ is a free R -module for any ring R .

Proof: (1) We have proved in [P-S-1] (Lemma 2.2 and Theorem 2.3) that if G is abelian then $\mathcal{S}(G; \mathbb{Z})$ is a free \mathbb{Z} -module. Moreover, Fact 2.6(4) implies that $\mathcal{S}(G; R) = \mathcal{S}(G; \mathbb{Z}) \otimes R$. Therefore $\mathcal{S}(G; R)$ is a free R -module.

(3) We will see in Section 4 (Fact 4.1) that the Kauffman bracket skein module of $F \times I$, where F is any surface, $I = [0, 1]$, is free for any ring of coefficients R and any $A^{\pm 1} \in R$. Therefore $\mathcal{S}(\pi_1(F); R) \simeq \mathcal{S}_{2,\infty}(F \times I; R, -1)$ is also free.

(2) It is a special case of (3).

(4) Let M be the complement of a $(2, 2k + 1)$ -torus knot in S^3 . D. Bullock proved in [B-5] that $\mathcal{S}_{2,\infty}(M; \mathbb{Z}[A^{\pm 1}], A)$ is a free $\mathbb{Z}[A^{\pm 1}]$ -module. Let us consider any ring R as a $\mathbb{Z}[A^{\pm 1}]$ -algebra determined by a homomorphism $f : \mathbb{Z}[A^{\pm 1}] \rightarrow R, f(1) = 1, f(A) = -1$. By Fact 2.7 and Theorem 2.8, $\mathcal{S}(\pi_1(M); R) \simeq \mathcal{S}_{2,\infty}(M; R, -1) \simeq \mathcal{S}_{2,\infty}(M; \mathbb{Z}[A^{\pm 1}], A) \otimes_{\mathbb{Z}[A^{\pm 1}]} R$ is a free R -module. \square

Theorems 3.2, 3.6 and Corollary 3.5 imply the following result.

Corollary 3.7

If G is abelian or if G is a free group or a fundamental group of a surface or the group of a $(2, 2k + 1)$ -torus knot then $TH_R(G)$ is a free R -module for any ring R .

If G is a free group than one can give an alternative prove that $TH(G)$ is a free R -module by using Proposition 9.1[B-H]. For the case $G = F_3, F_4$ see also comments on page 9 and Proposition 6.6(iii), [B-H].

It is worth mentioning that the skein modules $\mathcal{S}_{2,\infty}(M; R, A)$ have much more complicated structure than the modules $\mathcal{S}(\pi_1(M); R)$. For example, if $M = S^1 \times S^1 \times S^1$, then Theorems 2.8 and 3.6 imply that $\mathcal{S}_{2,\infty}(M; R, -1) \simeq \mathcal{S}(\pi_1(M); R) = \mathcal{S}(\mathbb{Z}^3; R)$ is a free R -module. We can prove, however, that the module $\mathcal{S}_{2,\infty}(M; \mathbb{Z}[A^{\pm 1}], A)$ has a big torsion part and, in particular, is not free.

Another interesting example was considered by J. Hoste and the first author in [H-P-3]. They proved that if M is a classical Whitehead manifold then the skein module $\mathcal{S}_{2,\infty}(M; \mathbb{Z}[A^{\pm 1}], A)$ is torsion free and infinitely generated. On the other hand Theorem 2.8 and the definition of skein algebra imply that $\mathcal{S}_{2,\infty}(M; R, -1) \simeq \mathcal{S}(\{e\}; R) \simeq R$ for any ring R .

4 On skein algebras of surfaces

Let F be an oriented surface and I be the closed interval $[0, 1]$. In this section we are investigating skein modules $\mathcal{S}_{2,\infty}(F \times I; R, A)$. They admit a natural operation of multiplication which can be defined in the following way.

For any two framed links L_1, L_2 in $F \times I$ we define $L_1 \cdot L_2$ to be a union of L_1 and L_2 , where L_1 is isotopically pushed into $F \times (1/2, 1)$ and L_2 is isotopically pushed into $F \times (0, 1/2)$. This multiplication is well defined on the set of ambient isotopy classes of links in $F \times I$ because if links L_i and L'_i are ambient isotopic in $F \times I$, for $i = 1, 2$, then $L_1 \cdot L_2$ is ambient isotopic to $L'_1 \cdot L'_2$. To show this, note that if links L and L' are ambient isotopic in $F \times I$ then we can choose an ambient isotopy between them which fixes the boundary of $F \times I$. Therefore, we can obtain an ambient isotopy between $L_1 \cdot L_2$ and $L'_1 \cdot L'_2$ by composing an ambient isotopy between L_1 and L'_1 in $F \times [1/2, 1]$, which fixes $\partial(F \times [1/2, 1])$, with an ambient isotopy between L_2 and L'_2 in $F \times [0, 1/2]$, which fixes $\partial(F \times [0, 1/2])$.

Having defined a product on the set of all ambient isotopy classes of framed, unoriented links in $F \times I$, $\mathcal{L}_{fr}(F \times I)$, we can linearly extend it to a multiplication in the module $R\mathcal{L}_{fr}(F \times I)$ and, finally, obtain a multiplication in $\mathcal{S}_{2,\infty}(F \times I; R, A)$. The element $[\emptyset] = 1$ is the identity in $\mathcal{S}_{2,\infty}(F \times I; R, A)$. If $A = -1$ then the multiplication defined above coincides with the multiplication considered in Sections 2 and 3.

Notice that the multiplication in $\mathcal{S}_{2,\infty}(F \times I; R, A)$ is uniquely determined by the surface F but it is not necessarily unique to the manifold $F \times I$. To illustrate this phenomenon let us consider a disc with two holes, F_0 , and a punctured torus, F_1 . Using Fact 4.1 one can easily prove that $\mathcal{S}_{2,\infty}(F_1 \times I; R, A)$ is a non-commutative algebra. On the other hand $\mathcal{S}_{2,\infty}(F_0 \times I; R, A)$ is commutative by Corollary 4.4. Hence the algebras $\mathcal{S}_{2,\infty}(F_1 \times I; R, A)$ and $\mathcal{S}_{2,\infty}(F_0 \times I; R, A)$ are clearly not isomorphic even though there is a homeomorphism of topological spaces $F_0 \times I \simeq F_1 \times I$.

In the first part of this section we prove a few basic properties of skein

algebras of surfaces. We show that $\mathcal{S}_{2,\infty}(F \times I; R, A)$ can be considered as an algebra over a ring of polynomials in many variables. We also describe $\mathcal{S}_{2,\infty}(F \times I; R, A)$ for F being a d -punctured sphere, where $d = 0, 1, 2, 3$. Moreover, we announce two more sophisticated results on skein algebras $\mathcal{S}_{2,\infty}(F \times I; R, A)$. We claim that if R has no zero divisors then the algebra $\mathcal{S}_{2,\infty}(F \times I; R, A)$ has no zero divisors either. Moreover, $\mathcal{S}_{2,\infty}(F \times I; R, A)$ considered as an algebra over a ring of polynomials is a central algebra. These two results require relatively long proofs based on Dehn's theorem classifying curves on surfaces. For that reason we decided to publish proofs of these results in a separate paper, [P-S-2].

In this section we assume that F is an oriented, compact, connected 2-manifold. We identify F with $F \times \{1/2\} \subset F \times I$. Let $L(F)$ be the set of all unoriented links in F without (homotopically) trivial components. We assume, as usual, that $\emptyset \in L(F)$. Notice that any link in $L(F)$ has a natural framing i.e. the framing parallel to F . Therefore we can consider elements of $L(F)$ as framed links in $F \times I$. Let $\mathcal{B}(F)$ be a set containing exactly one link from each class of ambient isotopic links in $L(F)$. We start with the following basic result (See [P-1], [H-P-2]).

Fact 4.1

For any surface F , any ring R , and any element $A^{\pm 1} \in R$ the skein module $\mathcal{S}_{2,\infty}(F \times I; R, A)$ is a free R -module with a basis $\{[L]\}_{L \in \mathcal{B}(F)}$.

For any surface $F \not\simeq S^1 \times I$ we define $d(F)$ to be the number of boundary components of F . For $F = S^1 \times I$ we set $d(F) = 1$.

Lemma 4.2

Let $F \not\simeq S^1 \times I$ be a surface with d boundary components denoted by $\partial_1 F, \partial_2 F, \dots, \partial_d F$. Let N_i , for $i \in \{1, 2, \dots, d\}$, be a regular neighborhood of $\partial_i F$. We assume that N_1, N_2, \dots, N_d are disjoint. Let K_i be a knot in $N_i(F) \subset F$ parallel to $\partial_i F$. For $F \simeq S^1 \times I$ set $d = 1$, $N_1 = F$ and define K_1 to be a knot in F parallel to ∂F . Then

1. *There is a natural isomorphism of R -algebras*

$$f : R[\partial_1, \partial_2, \dots, \partial_d] \rightarrow \mathcal{S}_{2,\infty}(N_1 \times I \cup N_2 \times I \cup \dots \cup N_d \times I; R, A)$$

such that $f(\partial_i) = [K_i]$.

2. The embedding $i : N_1 \times I \cup N_2 \times I \cup \dots \cup N_d \times I \rightarrow F \times I$ induces a monomorphism $i_* : R[\partial_1, \partial_2, \dots, \partial_d] \rightarrow \mathcal{S}_{2,\infty}(F \times I; R, A)$.

Proof:

1. Notice that there is an isomorphism between $\mathcal{S}_{2,\infty}(N_1 \times I \cup N_2 \times I \cup \dots \cup N_d \times I; R, A)$ and $\mathcal{S}_{2,\infty}(N_1 \times I; R, A) \otimes \mathcal{S}_{2,\infty}(N_2 \times I; R, A) \otimes \dots \otimes \mathcal{S}_{2,\infty}(N_d \times I; R, A)$ matching elements of the form

$$[L_1 \cup L_2 \cup \dots \cup L_d] \in \mathcal{S}_{2,\infty}(N_1 \times I \cup N_2 \times I \cup \dots \cup N_d \times I; R, A),$$

where L_i is a link in $N_i, i \in \{1, 2, \dots, d\}$, with elements

$$[L_1] \otimes [L_2] \otimes \dots \otimes [L_d] \in \mathcal{S}_{2,\infty}(N_1 \times I; R, A) \otimes \dots \otimes \mathcal{S}_{2,\infty}(N_d \times I; R, A).$$

Since $N_i \simeq S^1 \times I$, we can assume that $\mathcal{B}(N_i) = \{K_i, K_i^2, K_i^3, \dots\}$, where K_i^n denotes a link in N_i composed of n parallel copies of K_i . By Fact 4.1 there is an isomorphism of R -modules $f_i : R[\partial_i] \rightarrow \mathcal{S}_{2,\infty}(N_i \times I)$ such that $f_i(\partial_i^n) = [K_i^n]$. One can easily see that f_i is also an isomorphism of R -algebras. Therefore

$$f = f_1 \otimes f_2 \otimes \dots \otimes f_n : R[\partial_1, \partial_2, \dots, \partial_d] = R[\partial_1] \otimes R[\partial_2] \otimes \dots \otimes R[\partial_d] \rightarrow$$

$$\mathcal{S}_{2,\infty}(N_1 \times I; R, A) \otimes \mathcal{S}_{2,\infty}(N_2 \times I; R, A) \otimes \dots \otimes \mathcal{S}_{2,\infty}(N_d \times I; R, A) \simeq$$

$$\mathcal{S}_{2,\infty}(N_1 \times I \cup N_2 \times I \cup \dots \cup N_d \times I; R, A)$$

is an isomorphism.

2. We have defined K_1, \dots, K_d in such a way that any two links $K_1^{n_1} \cup K_2^{n_2} \cup \dots \cup K_d^{n_d}$ and $K_1^{n'_1} \cup K_2^{n'_2} \cup \dots \cup K_d^{n'_d}$ are ambient isotopic in $F \times I$ if and only if $(n_1, n_2, \dots, n_d) = (n'_1, n'_2, \dots, n'_d)$. Therefore we can assume that all links of the form $K_1^{n_1} \cup K_2^{n_2} \cup \dots \cup K_d^{n_d}$, for any $n_1, n_2, \dots, n_d \in \{0, 1, 2, \dots\}$ belong to the set $\mathcal{B}(F)$.

Let us consider a basis of $R[\partial_1, \partial_2, \dots, \partial_d]$ composed of all monomials. Notice that $i_*(\partial_1^{n_1} \partial_2^{n_2} \dots \partial_d^{n_d}) = [K_1^{n_1} \cup K_2^{n_2} \cup \dots \cup K_d^{n_d}] \in \mathcal{S}_{2,\infty}(F \times I; R, A)$. Therefore i_* carries different elements of the basis of $R[\partial_1, \partial_2, \dots, \partial_d]$ to different elements of the basis $\mathcal{B}(F)$ of the module $\mathcal{S}_{2,\infty}(F \times I; R, A)$. Hence i_* is a monomorphism.

□

Let $\mathcal{B}'(F)$ be a set of all links in $\mathcal{B}(F)$ without components parallel to the boundary of F . Notice that $\emptyset \in \mathcal{B}'(F)$. Fact 4.1 and Lemma 4.2 imply the following corollary.

Corollary 4.3

$\mathcal{S}_{2,\infty}(F \times I; R, A)$ considered as an $R[\partial_1, \partial_2, \dots, \partial_{d(F)}]$ -algebra has a basis $\{[L]\}_{L \in \mathcal{B}'(F)}$.

The next corollary describes the skein algebra $\mathcal{S}_{2,\infty}(F \times I; R, A)$ for a few important surfaces F (Compare [B-P]).

Corollary 4.4

Let F be a sphere or a disc or an annulus or a disc with two holes. Then $i_* : R[\partial_1, \dots, \partial_{d(F)}] \rightarrow \mathcal{S}_{2,\infty}(F \times I; R, A)$ is an isomorphism of R -algebras.

Proof: Notice that $\mathcal{B}'(F) = \{\emptyset\}$ for surfaces F listed in Corollary 4.4. Therefore the statement of Corollary 4.4 is implied by Corollary 4.3. □

If we combine Corollary 4.4 with Theorem 2.8 we will get the following result.

Corollary 4.5

The following homomorphisms

1. $f_1 : R[\partial_1] \rightarrow \mathcal{S}(\mathbb{Z}; R), f_1(\partial_1) = [1]$
2. $f_2 : R[\partial_1, \partial_2, \partial_3] \rightarrow \mathcal{S}(F_2; R), f_2(\partial_1) = [a], f_2(\partial_2) = [ab], f_2(\partial_3) = [b]$

are isomorphisms of R -algebras.

Proofs of the following two crucial results about skein algebras of surfaces will appear in our paper [P-S-2].

Theorem 4.6

Let $A \in R$ be an invertible element which is not a root of unity in R . Then $\mathcal{S}_{2,\infty}(F \times I; R, A)$ is a central $R[\partial_1, \dots, \partial_{d(F)}]$ -algebra i.e. $i_*(R[\partial_1, \dots, \partial_{d(F)}])$ is the center of the algebra $\mathcal{S}_{2,\infty}(F \times I; R, A)$.

Theorem 4.7

Let R be a ring without zero divisors. Suppose that

1. F is an orientable surface, $M = F \times I$, and $A^{\pm 1} \in R$; or
2. F is an unorientable surface of an even, negative Euler characteristic, M is a twisted I bundle over F , and $A = \pm 1$.

Then the skein algebra $\mathcal{S}_{2,\infty}(F \times I; R, A)$ has no zero divisors.

The above theorem is not true for a Klein bottle, KB . One can check that if $\frac{1}{2} \in R$ then $\mathcal{S}_{2,\infty}(KB \times I; R, -1) \simeq \mathcal{S}(\langle a, b \mid aba^{-1}b \rangle, R) \simeq R[x, y]/(x(y^2 - 4))$ has zero divisors.

We will observe in Section 7 that Theorem 4.7 has important applications to the theory of $Sl_2(\mathbb{C})$ -character varieties.

5 Relative skein algebras

In this section we will define relative skein modules of 3-manifolds. The relative skein module of $F \times I$, where F is an oriented surface, admits an algebra structure and therefore it is called a relative skein algebra of F . We will prove that the Brumfiel and Hilden algebra $H(G)$, for $G = \pi_1(F)$, is isomorphic to a relative skein algebra of F .

Let M be an oriented 3-manifold with a boundary and let $\gamma_0, \gamma_1 : [0, 1] \rightarrow \partial M$ be two fixed simple arcs in the boundary of M . By a special framed arc in M we mean an embedding $\iota : [0, 1] \times I \rightarrow M$ such that $\iota(s, 0) = \gamma_0(s) \in \partial M$, $\iota(s, 1) = \gamma_1(s) \in \partial M$, for any $s \in [0, 1]$. We also assume that $\iota(s, t)$ lies in the interior of M for $t \in (0, 1)$. We can consider a special framed arc as a ribbon inside of M whose ends lie exactly on $\gamma_0([0, 1])$ and $\gamma_1([0, 1])$.

We define a relative framed link in M to be a disjoint union of a special framed arc with a framed unoriented link lying in the interior of M . Since we consider \emptyset as a link, any special framed arc in M is also a relative framed link. We say that two relative framed links L and L' are ambient isotopic if there is an ambient isotopy of M which carries L to L' and is fixed on ∂M .

Let $\mathcal{L}_{fr}^{rel}(M)$ denote the set of all ambient isotopy classes of relative framed links in M . We define a relative skein module of M in the same way as it was done in Definition 2.1 but we replace $\mathcal{L}_{fr}(M)$ by $\mathcal{L}_{fr}^{rel}(M)$.

Definition 5.1

Let M be a 3-manifold and let R be any ring with a specified invertible element A . Let $\mathcal{S}_{2,\infty}^{rel}$ be the submodule of $R\mathcal{L}_{fr}^{rel}(M)$ generated by

1. skein expressions $L_+ - AL_0 - A^{-1}L_\infty$ where L_+, L_0, L_∞ are any skein related framed relative links in M ³.
2. $L \cup \bigcirc + (A^2 + A^{-2}) \cdot L$ for any relative framed link L in M .

We define the relative Kauffman bracket skein module of M as the quotient

$$\mathcal{S}_{2,\infty}^{rel}(M; R, A) = R\mathcal{L}_{fr}^{rel}(M)/S_{2,\infty}^{rel}.$$

The definition of a relative framed link in M , as well as the definition of a relative skein module of M , depends on a particular choice of simple arcs $\gamma_0, \gamma_1 : [0, 1] \rightarrow \partial M$. One can easily see, however, that different choices of arcs γ_0, γ_1 give isomorphic relative skein modules.

In this section we will be interested in the case when $M = F \times I$, for a surface F . We will notice that there is a natural product on $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$ similar to that on $\mathcal{S}_{2,\infty}(F \times I; R, A)$. However the relative skein modules of cylinders over surfaces, $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$, are usually more complicated than the classical skein modules, $\mathcal{S}_{2,\infty}(F \times I; R, A)$. We are able to describe them only for surfaces with boundaries. Our description is analogous to that given in Fact 4.1

As before we identify F with $F \times \{1/2\} \subset F \times I$. Let us assume that $\gamma_0, \gamma_1 : [0, 1] \rightarrow \partial F \subset \partial(F \times I)$ are two disjoint arcs lying in the same component of ∂F . Let $p_0 = \gamma_0(1/2), p_1 = \gamma_1(1/2)$. By a relative (unframed) link in F we will mean a disjoint union $L = L_0 \cup K$, where L_0 is a link in the interior of F and $K : [0, 1] \rightarrow F$ is an arc, $K(0) = p_0, K(1) = p_1$, and $K(t)$ lies in the interior of F , for $t \in (0, 1)$. We denote the set of all relative links in F without (homotopically) trivial components by $L^{rel}(F)$. We assume that $\emptyset \in L^{rel}(F)$. Let $\mathcal{B}^{rel}(F)$ be a set containing exactly one link from each class of ambient isotopic relative links in $L^{rel}(F)$.

Notice that a regular neighborhood in F of any relative link $L = L_0 \cup K \in L^{rel}(F)$ is a framed relative link in $F \times I$. Therefore one can identify elements of $\mathcal{B}^{rel}(F)$ with framed relative links in $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$.

Theorem 5.2

Let F be a surface with a boundary. Then $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$ is a free R -module with a basis $\{[L]\}_{L \in \mathcal{B}^{rel}(F)}$.

³Skein related links were introduced in Definition 2.1.

Sketch of proof: Let L be a relative framed link in $F \times I$. Then L can be represented by a diagram D in F . Notice that if we smooth each of the crossings of D in two possible ways, we will obtain a collection of relative links in F . This implies that L can be represented by a linear combination of links $\sum r_i [L_i]$, where $L_i \in \mathcal{B}^{rel}(F)$, $r_i \in R$.

Two diagrams, $D_1, D_2 \subset F$ represent the same relative framed link $L \subset F \times I$ if one can be transformed into the other by a sequence of Reidemeister moves. One can check that these moves do not change the linear combination $\sum r_i [L_i]$ assigned to L . Therefore, there is a homomorphism of R -modules

$$f_0 : R\mathcal{L}_{fr}^{rel}(F \times I) \rightarrow R\mathcal{B}^{rel}(F),$$

such that $f_0(L) = \sum r_i [L_i]$. One can show that this homomorphism yields an isomorphism $f_1 : \mathcal{S}_{2,\infty}^{rel}(F \times I; R, A) \rightarrow R\mathcal{B}^{rel}(F)$. \square

Theorem 5.2 is not true for closed surfaces. Let $F = S^2$ and let L be any framed arc in $F \times I$ whose ends lie on $\gamma([0, 1]) \times \{0\}$ and $\gamma([0, 1]) \times \{1\}$, for some curve $\gamma : [0, 1] \rightarrow F$. Then one can show that $[L] \neq 0$ in $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$ but $(A^6 - 1)[L] = 0$. Therefore $\mathcal{S}_{2,\infty}^{rel}(S^2 \times I; R, A)$ is not free.

In order to define a multiplication operation on $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$ it is convenient to consider arcs γ_0, γ_1 defined as follows. Let $\gamma : [0, 1] \rightarrow F$ be a simple arc in the interior of F and let $\gamma_0, \gamma_1 \in F \times I$ be given by $\gamma_0(t) = (\gamma(t), 0)$, $\gamma_1(t) = (\gamma(t), 1)$, for any $t \in [0, 1]$. Let $h_1 : F \times [0, 1] \rightarrow F \times [\frac{1}{2}, 1]$, $h_2 : F \times [0, 1] \rightarrow F \times [0, \frac{1}{2}]$ be maps given by $h_1(x, t) = (x, \frac{1}{2} + \frac{1}{2}t)$, $h_2(x, t) = (x, \frac{1}{2}t)$ for $x \in F, t \in [0, 1]$. We define the product of relative framed links $L_1, L_2 \subset F \times I$, to be $L_1 \cdot L_2 = h_1(L_1) \cup h_2(L_2) \subset F \times [\frac{1}{2}, 1] \cup F \times [0, \frac{1}{2}] = F \times [0, 1]$. One can easily see that $h_1(L_1) \cup h_2(L_2)$ is a relative framed link in $F \times I$. Moreover, we can uniquely extend the above multiplication to a multiplication operation on $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$. This multiplication will be associative but (in general) not commutative. The module $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$ considered together with this multiplication will be called the relative skein algebra of a surface F . Whenever we will talk about the relative skein algebras of surfaces we will always assume that the arcs γ_0, γ_1 are defined as above. Notice that the any choice of $\gamma : [0, 1] \rightarrow F$ gives the same (up to an isomorphism) algebra structure on $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$.

The next result shows that the relative skein algebra of F can be considered as a generalization (or a deformation) of the algebra $H_R(G)$, $G = \pi_1(F)$, introduced in Section 3.

Theorem 5.3

Let F be any oriented surface and let R be a ring. Then $H_R(\pi_1(F))$ is isomorphic as an R -algebra to $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1)$.

Proof: Let us fix a point $p \in F$ and assume that $F = F \times \{1/2\} \subset F \times I$, $I = [0, 1]$. Let $G = \pi_1(F, p)$. We will define a homomorphism $f_0 : RG \rightarrow \mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1)$ so that f_0 vanishes on $I = (h(g + g^{-1}) - (g + g^{-1})h) \triangleleft RG$, and thus factors through $H_R(G)$.

In the proof we will consider relative (unframed) links in $F \times I$. A relative (unframed) link in $F \times I$ is a disjoint union of a (possibly empty) unoriented link in $F \times I$ with an arc in $F \times I$ joining $(p, 0)$ with $(p, 1)$. We denote the set of all ambient isotopy classes of relative links in $F \times I$ by $\mathcal{L}^{rel}(F \times I)$.

Notice that $[L] \in \mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1)$ does not depend on the framing of a relative link L . Therefore we can define $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1)$ using unframed links, $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1) = R\mathcal{L}^{rel}(F \times I)/\mathcal{S}^{rel}$, where \mathcal{S}^{rel} is a submodule of $R\mathcal{L}^{rel}(F \times I)$ generated by expressions

1. $L_+ + L_0 + L_\infty$, for any skein related (unframed) relative links L_+, L_0, L_∞ in $F \times I$.
2. $L \cup \bigcirc + 2 \cdot L$, for any (unframed) relative link in L in $F \times I$.

Suppose that $g \in G = \pi_1(F, p)$ is the homotopy class of an oriented loop $\xi : [0, 1] \rightarrow F, \xi(0) = \xi(1) = p$. Then we define an arc $K_g : [0, 1] \rightarrow F \times I, K_g(t) = (\xi(t), 1 - t)$, $t \in [0, 1]$. Notice that K_e is the trivial arc, $K_e : [0, 1] \rightarrow F \times I, K_e(t) = (p, 1 - t)$.

Let $f_0 : RG \rightarrow \mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1)$ be an R -module homomorphism such that $f_0(g) = [K_g] \in \mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1)$. Notice that $f_0(g_1 g_2) = [K_{g_1 g_2}] = [K_{g_1}][K_{g_2}] = f_0(g_1)f_0(g_2)$. Therefore f_0 is a homomorphism of R -algebras.

Let $L_+ = K_e \cup K$, where K_e is the trivial arc and K is a knot in $F \times I$ representing (with a given orientation) the conjugacy class of some $g \in G$. Homotopic relative framed links in $F \times I$ are identified in $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1)$. Therefore we can place K as in Fig. 2. Then $[L_+] = -[L_0] - [L_\infty]$ in $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1)$, where $L_0 = K_g, L_\infty = K_{g^{-1}}$ are presented in Figure 2.

Hence $[L_+] = -f_0(g + g^{-1})$. Notice that $[L_+]$ commutes with any relative link in $F \times [0, 1]$. Therefore $f_0(h(g + g^{-1})) = f_0((g + g^{-1})h)$ for any $g, h \in G$ and f_0 vanishes on $I = (h(g + g^{-1}) - (g + g^{-1})h) \triangleleft RG$. Hence f_0 induces a homomorphism $f_1 : H_R(G) \rightarrow \mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1)$.

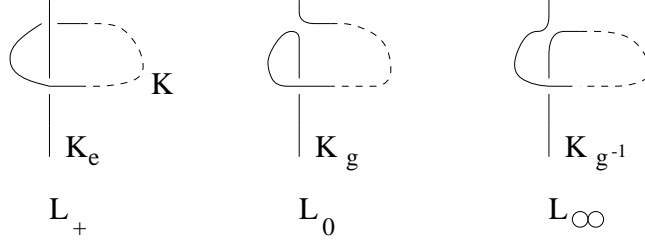


Figure 2

Lemma 5.4

f_1 is an epimorphism.

Proof: Notice that for any relative link L in $F \times I$, which is composed of an arc K_g and knots L_1, L_2, \dots, L_n , $L = K_g \cup L_1 \cup L_2 \cup \dots \cup L_n$, we have $[L] = [K_g][L_1 \cup K_e] \cdot [L_2 \cup K_e] \cdot \dots \cdot [L_n \cup K_e] \in \mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1)$. Let L_i with a given orientation correspond to the conjugacy class of some $g_i \in G$. Then, as we have seen before, $[L_i \cup K_e] = -f_1(g_i + g_i^{-1})$. Hence $[L_i \cup K_e] \in \text{Im } f_1$. Since $[K_g] = f_1(g)$ and f_1 is a homomorphism of algebras, $[L] \in \text{Im } f_1$. The elements $[L]$, where L are relative links, generate $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1)$. Therefore f_1 is an epimorphism. \square

We are going to complete the proof of Theorem 5.3 by showing that f_1 is a monomorphism.

Let $J \triangleleft RG \otimes_R \mathcal{S}(G; R)$ be an ideal generated by elements $g \otimes [h] - gh \otimes 1 - gh^{-1} \otimes 1$, for any $g, h \in G$. Let $\rho_0 : R\mathcal{L}^{rel}(F \times I) \rightarrow (RG \otimes_R \mathcal{S}(G; R))/J$ be a homomorphism of R -modules defined in the following way:

If L is a relative link in $F \times I$ of the form $L = K_g \cup L'$, where L' is a link disjoint from an arc K_g , then $\rho_0(L) = g \otimes \hat{\xi}([L']) + J$ ⁴.

Lemma 5.5

Let L_+, L_0, L_∞ be any skein related links in $F \times I$. Then $\rho_0(L_+ + L_0 + L_\infty) = 0$.

Proof: Let $L_+ = K_g \cup L'$, where $g \in G$ and L' is a link in $F \times I$. We will call the crossing in L_+ which has to be smoothed in order to obtain L_0 or L_∞ a *specified crossing* in L_+ . There are three possibilities:

1. The specified crossing in L_+ is between two connected components of L' or is a self-crossing of a component of L' .

⁴The homomorphism $\hat{\xi}$ was introduced in Theorem 2.8.

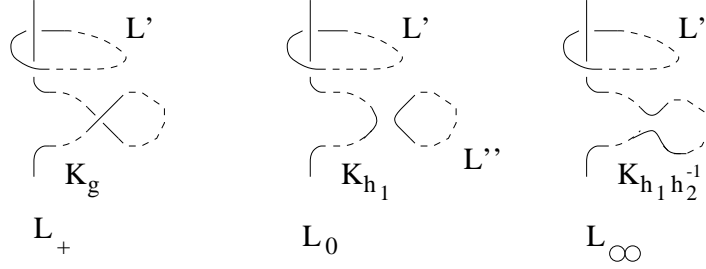


Figure 3

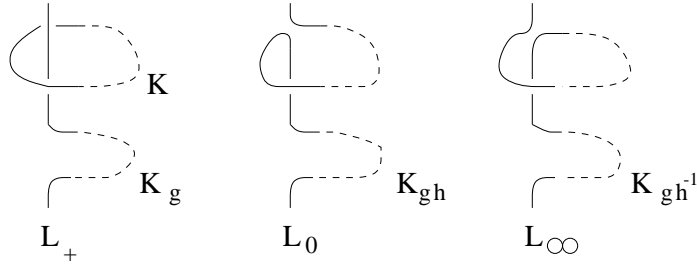


Figure 4

Then $L_0 = K_g \cup L'_0$, $L_\infty = K_g \cup L'_\infty$ and $\rho_0(L_+ + L_0 + L_\infty) = g \otimes \hat{\xi}([L']) + g \otimes \hat{\xi}([L'_0]) + g \otimes \hat{\xi}([L'_\infty]) + J = g \otimes \hat{\xi}([L'] + [L'_0] + [L'_\infty]) + J$. Since L', L'_0, L'_∞ are skein related, $[L'] + [L'_0] + [L'_\infty] = 0$ in $\mathcal{S}_{2,\infty}(F \times I; R, -1)$. Therefore $\rho_0(L_+ + L_0 + L_\infty) = 0$.

2. The specified crossing in L_+ is a self-intersection of K_g .

Then $L_0 = K_{h_1} \cup L' \cup L''$, $L_\infty = K_{h_1 h_2^{-1}} \cup L'$ have forms presented in Fig. 3, $h_1 h_2 = g$, and the knot L'' corresponds (with a given orientation) to the conjugacy class of $h_2 \in G$.

Therefore $\rho_0(L_+ + L_0 + L_\infty) = g \otimes \hat{\xi}([L']) + h_1 \otimes \hat{\xi}([L' \cup L'']) + h_1 h_2^{-1} \otimes \hat{\xi}([L']) + J = (g \otimes 1 + h_1 \otimes (-[h_2]) + h_1 h_2^{-1} \otimes 1 + J) \cdot (1 \otimes \hat{\xi}([L']) + J) = (g \otimes 1 - h_1 h_2 \otimes 1 - h_1 h_2^{-1} \otimes 1 + h_1 h_2^{-1} \otimes 1 + J) \cdot (1 \otimes \hat{\xi}([L']) + J) = 0$.

3. The specified crossing in L_+ is between K_g and a connected component K of L' .

Choose some orientation of K . Then K corresponds to the conjugacy class of some $h \in \pi_1(F, p) = G$. Let $L'' = L' \setminus K$. Then L_0 and L_∞ have forms $K_{gh} \cup L''$ and $K_{gh^{-1}} \cup L''$ as shown in Figure 4.

Therefore $\rho_0(L_+ + L_0 + L_\infty) = g \otimes \hat{\xi}([K][L'']) + gh \otimes \hat{\xi}([L'']) + gh^{-1} \otimes \hat{\xi}([L'']) + J = (g \otimes \hat{\xi}([K]) + gh \otimes 1 + gh^{-1} \otimes 1 + J) \cdot (1 \otimes \hat{\xi}([L'']) + J) = (g \otimes (-[h]) + gh \otimes 1 + gh^{-1} \otimes 1 + J)(1 \otimes \hat{\xi}([L'']) + J) = 0$ in $(RG \otimes_R \mathcal{S}(G; R))/J$.

□

Observe that if $L = K_g \cup L'$, where L' is a link $F \times I$, then $\rho_0(L \cup \bigcirc + 2L) = \rho_0(L \cup \bigcirc) + 2\rho_0(L) = g \otimes \hat{\xi}([L' \cup \bigcirc]) + 2g \otimes \hat{\xi}([L']) + J = g \otimes \hat{\xi}(-2[L']) + 2g \otimes \hat{\xi}([L']) + J = 0$.

The above observation and Lemma 5.5 imply that $\rho_0(\mathcal{S}^{rel}) = 0$, and hence, there is a homomorphism $\rho_1 : \mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1) \rightarrow (RG \otimes_R \mathcal{S}(G; R))/J$, such that $\rho_1([L]) = g \otimes \hat{\xi}([L']) + J$ for any relative framed link $L = K_g \cup L'$.

Lemma 5.6

There is a homomorphism $\mu : (RG \otimes_R \mathcal{S}(G; R))/J \rightarrow H_R(G)$, such that $\mu(g \otimes 1 + J) = [g] \in H_R(G)$, for any $g \in G$.

Proof: Let $\mu_0 : RG \otimes \mathcal{S}(G; R) \rightarrow H_R(G)$ be a homomorphism of R -modules $\mu_0(x \otimes y) = [x] \cdot \psi(y)$, for any $x \in RG, y \in \mathcal{S}(G; R)$ ⁵.

Let $x_1, x_2 \in RG, y_1, y_2 \in \mathcal{S}(G; R)$. Then $\mu_0((x_1 \otimes y_1)(x_2 \otimes y_2)) = \mu_0((x_1 x_2) \otimes (y_1 y_2)) = [x_1 x_2] \psi(y_1 y_2) = [x_1][x_2] \psi(y_1) \psi(y_2)$. Since $\psi(y_1) \in TH_R(G) \subset \text{Center } H_R(G)$, we have $\mu_0((x_1 \otimes y_1)(x_2 \otimes y_2)) = x_1 \psi(y_1) x_2 \psi(y_2) = \mu_0(x_1 \otimes y_1) \mu_0(x_2 \otimes y_2)$. Therefore μ_0 is a homomorphism of R -algebras. Moreover, $\mu_0(g \otimes [h] - gh \otimes 1 - gh^{-1} \otimes 1) = [g]([h] + [h^{-1}]) - [gh] - [gh^{-1}] = 0$ for any $g, h \in G$. Hence μ_0 yields $\mu : (RG \otimes_R \mathcal{S}(G; R))/J \rightarrow H_R(G)$, such that $\mu(g \otimes 1 + J) = [g]$. □

Notice that $\mu \circ \rho_1 \circ f_1 : H_R(G) \rightarrow H_R(G)$. Let $g \in G$. Then $\mu \circ \rho_1 \circ f_1([g]) = \mu \circ \rho_1([K_g]) = \mu(g \otimes 1 + J) = [g]$. Since $\{[g]\}_{g \in G}$ generate $H_R(G)$, $\mu \circ \rho_1 \circ f_1 = \text{id}_{H_R(G)}$. Therefore f_1 is a monomorphism, and the proof of Theorem 5.3 has been completed. □

Under some additional conditions we can strengthen the statement of Theorem 5.3 in the following way.

Suppose that $A^{\pm 1} \in R$ and there is an involution τ_0 on R such that $\tau_0(A) = A^{-1}$. For example

- $A = -1$ and $\tau_0 : R \rightarrow R, \tau_0 = \text{id};$ or

⁵The homomorphism ψ was introduced in Theorem 3.2.

- $R = R_0[A^{\pm 1}]$ for some ring R_0 and $\tau_0 : R \rightarrow R$, $\tau_0|_{R_0} = id_{R_0}$, $\tau_0(A) = A^{-1}$.

Then we can define an involution τ on $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$ in the following way.

Let $\tau_1 : R\mathcal{L}_{fr}^{rel}(F \times I) \rightarrow R\mathcal{L}_{fr}^{rel}(F \times I)$ be an additive function such that $\tau_1(rL) = \tau_0(r)h(L)$, where $h : F \times I \rightarrow F \times I$ is a homeomorphism $h(x, t) = (h, 1 - t)$, $x \in F, t \in [0, 1]$. Then one can easily check that $\tau_1(\mathcal{S}_{fr}^{rel}) = \mathcal{S}_{fr}^{rel}$. Therefore τ_1 yields an additive function $\tau : \mathcal{S}_{2,\infty}^{rel}(F \times I; R, A) \rightarrow \mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$. One can further check that the following fact holds.

Fact 5.7

τ is an involution on $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$, i.e. τ is an anti-isomorphism of $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$ and $\tau \circ \tau = id$. Moreover, the following diagram commutes:

$$\begin{array}{ccccc} R & \times & \mathcal{S}_{2,\infty}^{rel}(F \times I; R, A) & \rightarrow & \mathcal{S}_{2,\infty}^{rel}(F \times I; R, A) \\ \downarrow \tau_0 & & \downarrow \tau & & \downarrow \tau \\ R & \times & \mathcal{S}_{2,\infty}^{rel}(F \times I; R, A) & \rightarrow & \mathcal{S}_{2,\infty}^{rel}(F \times I; R, A) \end{array}$$

Recall that we defined an involution ι on the algebra $H_R(G)$ at the beginning of Section 3.

Theorem 5.8

Let R be a commutative ring with the trivial involution $\tau_0 = id_R$. Then for any surface F there exists a preserving involutions isomorphism between $H_R(\pi_1(F))$ and $\mathcal{S}_{2,\infty}^{rel}(F \times I; R, -1)$.

Proof: Notice that for any $g \in \pi_1(F, p)$ $f_1(\iota([g])) = f_1([g^{-1}]) = [K_{g^{-1}}] = \tau([K_g])$. Therefore f_1 is an isomorphism of R -algebras which preserves involutions. \square

We end this section with a theorem which can be considered as a generalization of Theorem 3.2. Notice that if $\frac{1}{2} \in R$ then Theorem 3.2 (together with Theorems 5.3 and 2.8) implies that there exist an injection $\mathcal{S}_{2,\infty}(F \times I; R, A) \rightarrow \mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$ for $A = -1$. This fact turns out to be true for any A , and it has a simple topological interpretation.

In order to introduce it consider a surface F with a boundary and a disc $D \subset F$ such that $D \cap \partial F$ is an arc in ∂F . Let $\gamma : [0, 1] \rightarrow D \subset F$ be an arc and let K_0 be the trivial special framed arc in $F \times I$ associated with γ , that is, $K_0 : [0, 1] \times I \rightarrow F \times I$, $K(s, t) = (\gamma(s), 1 - t)$. Since any framed link L in

$F \times I$ can be pushed inside $(F \setminus D) \times I$ in a unique way (up to an ambient isotopy in $(F \setminus D) \times I$), there is a well defined operation of making a disjoint union of L with K_0 .

Theorem 5.9

Let F be a surface, $\partial F \neq \emptyset$, and let a disjoint union, $L \cup K_0 \subset F \times I$, be defined as above for any framed link $L \subset F \times I$. Then there exists a homomorphism of algebras $\eta : \mathcal{S}_{2,\infty}(F \times I; R, A) \rightarrow \mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$ such that $\eta([L]) = [L \cup K_0]$, for any framed link L in $F \times I$.

Moreover, if $A^2 + A^{-2}$ is invertible in R then $\eta : \mathcal{S}_{2,\infty}(F \times I; R, A) \rightarrow \mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$ is a monomorphism.

Proof: Let $\eta_0 : R\mathcal{L}_{fr}(F \times I) \rightarrow R\mathcal{L}_{fr}^{rel}(F \times I)$ be a homomorphism of R -modules such that $\eta_0(L) = L \cup K_0$ for any link L in $F \times I$. Notice that this homomorphism induces a homomorphism of skein modules $\eta : \mathcal{S}_{2,\infty}(F \times I; R, A) \rightarrow \mathcal{S}_{2,\infty}^{rel}(F \times I; R, A)$. Moreover, η is a homomorphism of rings.

Suppose now that $A^2 + A^{-2}$ is invertible in R . We will show that η is a monomorphism. Let K_1 be a framed arc in $D \times I \subset F \times I$ such that $K_0 \cap K_1 = \gamma([0, 1]) \times \{0\} \cup \gamma([0, 1]) \times \{1\}$ and $K_0 \cup K_1$, after pushing it inside the interior of $D \times I$, is a trivial framed knot in $F \times I$. Then for any relative framed link L in $F \times I$, $L \cup K_1$ is a framed link which can be isotoped into the interior of $F \times I$. Moreover, we have a homomorphism of modules $\nu : \mathcal{S}_{2,\infty}^{rel}(F \times I; R, A) \rightarrow \mathcal{S}_{2,\infty}(F \times I; R, A)$ such that $\nu([L]) = [L \cup K_1]$. Notice that $\nu \circ \eta : \mathcal{S}_{2,\infty}(F \times I; R, A) \rightarrow \mathcal{S}_{2,\infty}(F \times I; R, A)$, $\nu \circ \eta([L]) = \nu([L \cup K_0]) = [L \cup K_0 \cup K_1] = [L \cup \bigcirc] = -(A^2 + A^{-2})[L]$. Therefore if $A^2 + A^{-2}$ is invertible in R then ν is a monomorphism. \square

6 Estimating minimal numbers of generators of skein algebras; Part 1

In this section we will find minimal numbers of generators of skein algebras of surfaces with coefficients in $\mathbb{Z}[A^{\pm 1}]$. We will also calculate minimal numbers of generators of skein algebras of abelian and non-abelian free groups.

We start with a fact proven in [B-3] and concerning algebras $\mathcal{S}_{2,\infty}(F \times I; \mathbb{Z}[A^{\pm 1}], A)$ discussed in Section 4.

Let $F_{g,n}$ be a surface of genus g with n boundary components. If $n \geq 1$ then we can present $F_{g,n}$ a disc with $2g + n - 1$ handles numbered as shown in Fig. 5.

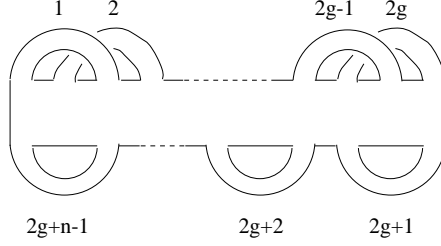


Figure 5: Surface $F_{g,n}$

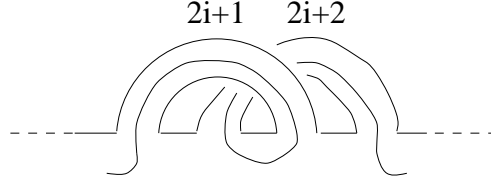


Figure 6

Let $S \subset \{1, 2, \dots, 2g + n - 1\}$. The standard knot of type S in $F_{g,n}$, $n \geq 1$, denoted by K_S , is a knot which satisfies the following conditions:

1. if $i \in S$ then K_S meets i -th handle exactly once.
2. if $i \notin S$ then K_S does not meet i -th handle.
3. if K_S meets two overlapping handles (i.e. $2i + 1, 2i + 2 \in S$, for some $i \in \{0, 1, \dots, g - 1\}$) then it does it in a way shown in Fig. 6.

We assume that $F_{g,1} \subset F_{g,0}$ and that the standard knots, K_S , in $F_{g,1}$ are also the standard knots in $F_{g,0}$. Therefore each surface F has $2^{\text{rank } H_1(F)}$ standard knots.

Theorem 6.1 ([B-3])

$\mathcal{S}_{2,\infty}(F_{g,n} \times I; \mathbb{Z}[A^{\pm 1}], A)$ is a $\mathbb{Z}[A^{\pm 1}]$ -algebra generated by $2^N - 1$ elements $[K_S]$, $N = \text{rank } H_1(F)$, for all possible subsets $S \subset \{1, 2, \dots, N\}$, $S \neq \emptyset$.

Using Bullock's proof of Theorem 6.1, we can prove the following theorem.

Theorem 6.2

Let M be a compact manifold of a Heegaard genus g . Then there is a link L in M of $N = 2^g - 1$ components, K_1, K_2, \dots, K_N , such that for any family of disjoint regular neighborhoods $K_i \subset V_i \simeq S^1 \times D^2$, $i = 1, \dots, N$, the embedding $i : V_1 \cup V_2 \cup \dots \cup V_N \rightarrow M$ induces an epimorphism of R -modules $i_* : \mathcal{S}_{2,\infty}(V_1; R, A) \otimes \mathcal{S}_{2,\infty}(V_2; R, A) \otimes \dots \otimes \mathcal{S}_{2,\infty}(V_N; R, A) \rightarrow \mathcal{S}_{2,\infty}(M; R, A)$.

Proof: Let us consider any linear order, $<$, on the set of non-empty subsets of $\{1, 2, \dots, N\}$, where $N = \text{rank} H_1(F)$. Bullock's proof of Theorem 6.1 can be modified to give a stronger version of Theorem 6.1. Namely, $\mathcal{S}_{2,\infty}(F_{g,n} \times I; \mathbb{Z}[A^{\pm 1}], A)$ is a $\mathbb{Z}[A^{\pm 1}]$ -module generated by monomials $[K_{S_1}] \cdot [K_{S_2}] \cdot \dots \cdot [K_{S_k}]$ such that $S_1 \leq S_2 \leq \dots \leq S_k$.

Suppose that M has a Heegaard splitting $M = H_1 \cup H_2$ of genus g . Then H_1 is homeomorphic to $F_{0,g+1} \times [0, 1]$. Let L be a link in $H_1 \simeq F_{0,g+1} \times [0, 1]$ composed of $2^g - 1$ knots K_S , where $S \subset \{1, \dots, g\}$, $S \neq \emptyset$, in such a way that K_{S_1} lies 'above' K_{S_2} in $H_1 \simeq F_{0,g+1} \times I$, iff $S_1 > S_2$. Notice that then the embedding $j : V_1 \cup V_2 \cup \dots \cup V_N \rightarrow H_1$ induces an epimorphism of R -modules

$$j_* : \mathcal{S}_{2,\infty}(V_1; R, A) \otimes \mathcal{S}_{2,\infty}(V_2; R, A) \otimes \dots \otimes \mathcal{S}_{2,\infty}(V_N; R, A) \rightarrow \mathcal{S}_{2,\infty}(H_1; R, A).$$

Moreover, since every link in M can be pushed inside H_1 we have an epimorphism $\mathcal{S}_{2,\infty}(H_1; \mathbb{Z}[A^{\pm 1}], A) \rightarrow \mathcal{S}_{2,\infty}(M \times I; \mathbb{Z}[A^{\pm 1}], A)$. The composition of this epimorphism with j_* gives the required epimorphism i_* . \square

Theorem 6.1 implies the following theorem.

Theorem 6.3

1. Let F_n denote a free group, $F_n = \langle g_1, g_2, \dots, g_n \rangle$. Then for any ring R the skein algebra $\mathcal{S}(F_n; R)$ is generated by $2^n - 1$ elements of the form $[g_{i_1} g_{i_2} \dots g_{i_k}]$, where $1 \leq i_1 < i_2 < \dots < i_k \leq n$ and $k \geq 1$.
2. For any ring R the skein algebra $\mathcal{S}(\mathbb{Z}^n; R)$ is generated by $2^n - 1$ elements $[(\epsilon_1, \dots, \epsilon_n)]$, where $(\epsilon_1, \dots, \epsilon_n) \in \mathbb{Z}^n$, $\epsilon_1, \epsilon_2, \dots, \epsilon_n \in \{0, 1\}$ and $\epsilon_i \neq 0$ for at least one i .
3. If $R = \mathbb{Z}$ then the numbers of generators given above are minimal, i.e. neither $\mathcal{S}(F_n; \mathbb{Z})$ nor $\mathcal{S}(\mathbb{Z}^n; \mathbb{Z})$ can have fewer generators than $2^n - 1$.⁶

Proof:

1. We can consider R as a $\mathbb{Z}[A^{\pm 1}]$ -algebra determined by the homomorphism $f : \mathbb{Z}[A^{\pm 1}] \rightarrow R$, $f(1) = 1$, $f(A) = -1$. Theorem 6.1 and Fact

⁶In fact we prove the following more general fact: If R is a ring such that \mathbb{Z}_2 can be an epimorphic image of R then neither $\mathcal{S}(F_n; R)$ nor $\mathcal{S}(\mathbb{Z}^n; R)$ can have fewer generators than $2^n - 1$.

2.7 implies that the algebra $\mathcal{S}_{2,\infty}(F_{0,n+1} \times I; R, -1)$ is generated by $[K_S]$, $S \subset \{1, 2, \dots, n\}, S \neq \emptyset$. Let us fix an isomorphism between $F_n = \langle g_1, g_2, \dots, g_n \rangle$ and $\pi_1(F_{0,n+1})$ in such a way that g_i corresponds to the homotopy class of $K_{\{i\}}$. Then, by Theorem 2.8, $\mathcal{S}_{2,\infty}(F_{0,n+1} \times I; R, -1)$ is isomorphic to $\mathcal{S}(F_n; R)$ and the generators $[K_S]$, for $S = \{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$, of the skein algebra $\mathcal{S}_{2,\infty}(F_{0,n+1} \times I; R, -1)$ correspond to elements $(-1)^k [g_{i_1} g_{i_2} \dots g_{i_k}] \in \mathcal{S}(F_n; R)$.

2. Let $f : F_n = \langle g_1, \dots, g_n \rangle \rightarrow \mathbb{Z}^n$ be an epimorphism given by $f(g_i) = (0, 0, \dots, 1, \dots, 0)$, where the single 1 stands at the i -th place. By Fact 2.6(3), f yields an epimorphism $f_* : \mathcal{S}(F_n; R) \rightarrow \mathcal{S}(\mathbb{Z}^n; R)$. Notice that $f_*([g_{i_1} g_{i_2} \dots g_{i_k}]) = [(\epsilon_1, \dots, \epsilon_n)]$, where $\epsilon_1, \epsilon_2, \dots, \epsilon_n \in \{0, 1\}$ and $\epsilon_j = 1$, for $j = 1, 2, \dots, n$, iff j is equal to one of the numbers i_1, i_2, \dots, i_k . Therefore $\mathcal{S}(\mathbb{Z}^n; R)$ is generated by all elements of the form $[(\epsilon_1, \dots, \epsilon_n)]$, where $(\epsilon_1, \dots, \epsilon_n) \in \mathbb{Z}^n$, $\epsilon_i \in \{0, 1\}$ and at least one ϵ_i is equal to 1.
3. Let $h : \mathbb{Z}^n \rightarrow \mathbb{Z}_2^n$ be a natural projection sending $(x_1, \dots, x_n) \in \mathbb{Z}^n$ to $(x_1 \bmod 2, \dots, x_n \bmod 2) \in \mathbb{Z}_2^n$. Let $f : F_n \rightarrow \mathbb{Z}^n$ be defined as in the previous paragraph. Since $h_* : \mathcal{S}(\mathbb{Z}^n; \mathbb{Z}) \rightarrow \mathcal{S}(\mathbb{Z}_2^n; \mathbb{Z})$ and $h_* \circ f_* : \mathcal{S}(F_n; \mathbb{Z}) \rightarrow \mathcal{S}(\mathbb{Z}_2^n; \mathbb{Z})$ are epimorphisms, it is enough to show that $\mathcal{S}(\mathbb{Z}_2^n; \mathbb{Z})$ requires at least $2^n - 1$ generators. Suppose that $\mathcal{S}(\mathbb{Z}_2^n; \mathbb{Z})$ can be generated by k elements. By Fact 2.6(4), $\mathcal{S}(\mathbb{Z}_2^n; \mathbb{Z}_2) = \mathcal{S}(\mathbb{Z}_2^n; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_2$. Therefore $\mathcal{S}(\mathbb{Z}_2^n; \mathbb{Z}_2)$ can also be generated by k elements. Let $g, h \in \mathbb{Z}_2^n$. Then $gh = gh^{-1}$ and $[g] \otimes [h] = [gh] + [gh^{-1}] = 2[gh] = 0$ in $\mathcal{S}(\mathbb{Z}_2^n; \mathbb{Z}_2)$ (We use the multiplicative notation for the addition in \mathbb{Z}_2^n). Moreover, $[e] = 0$ in $\mathcal{S}(\mathbb{Z}_2^n; \mathbb{Z}_2)$ (e is the identity in \mathbb{Z}_2^n). Therefore $\mathcal{S}(\mathbb{Z}_2^n; \mathbb{Z}_2)$ is isomorphic to the ring $R = \mathbb{Z}_2[x_g; g \in \mathbb{Z}_2^n] / (x_e, x_g x_h)$, where $\mathbb{Z}_2[x_g; g \in \mathbb{Z}_2^n]$ means the ring of polynomials in 2^n variables $x_g, g \in \mathbb{Z}_2^n$. We will show that the ring R requires at least $2^n - 1$ generators. Notice that for any $x, y \in R$ $xy = 0$ or $xy = 1$. Therefore R is a \mathbb{Z}_2 -linear space of dimension 2^n with a basis $\{x_g\}_{g \in \mathbb{Z}_2^n \setminus \{e\}} \cup \{1\}$. If y_1, y_2, \dots, y_k generate R then $1, y_1, y_2, \dots, y_k$ span R as a linear space (because $y_i y_j = 0$ or $y_i y_j = 1$). Therefore $k \geq 2^n - 1$.

□

Notice that Fact 2.6(3) implies the following corollary to Theorem 6.3.

Corollary 6.4

Let G be a group generated by $g_1, g_2, \dots, g_n \in G$. Then for any ring R the

skein algebra $\mathcal{S}(G; R)$ is generated by elements of the form $[g_{i_1}g_{i_2}\dots g_{i_k}]$, where $1 \leq i_1 < i_2 < \dots < i_k \leq n$ and $k \geq 1$.

Using Theorem 6.3 we can prove the following result.

Proposition 6.5

The algebra $\mathcal{S}_{2,\infty}(F_{g,n} \times I; \mathbb{Z}[A^{\pm 1}], A)$ cannot be generated by fewer than $2^N - 1$ elements, where $N = \text{rank} H_1(F)$.

Proof: By Fact 2.7

$$\mathcal{S}_{2,\infty}(F_{g,n} \times I; \mathbb{Z}, -1) \simeq \mathcal{S}_{2,\infty}(F_{g,n} \times I; \mathbb{Z}[A^{\pm 1}], A) \otimes_{\mathbb{Z}[A^{\pm 1}]} \mathbb{Z},$$

where \mathbb{Z} is considered as a $\mathbb{Z}[A^{\pm 1}]$ -algebra via homomorphism $h : \mathbb{Z}[A^{\pm 1}] \rightarrow \mathbb{Z}$, $h(A) = -1$. Let k be the minimal number of generators of $\mathcal{S}_{2,\infty}(F_{g,n} \times I; \mathbb{Z}[A^{\pm 1}], A)$. Then $\mathcal{S}_{2,\infty}(F_{g,n} \times I; \mathbb{Z}, -1) \simeq \mathcal{S}(\pi_1(F_{g,n}); \mathbb{Z})$ can also be generated by k elements. Let $f : \pi_1(F_{g,n}) \rightarrow \mathbb{Z}^N$ be the abelianizing homomorphism. Then $f_* : \mathcal{S}(\pi_1(F_{g,n}); \mathbb{Z}) \rightarrow \mathcal{S}(\mathbb{Z}^N; \mathbb{Z})$ is onto and by Theorem 6.3(3) $k \geq 2^N - 1$. \square

7 Characters of $SL_2(\mathbb{C})$ -representations of groups

In this section we combine the results of previous sections with the results of [B-H] and apply them to a study of connections between skein algebras and $SL_2(\mathbb{C})$ -character varieties. We start with a brief presentation of the approach to $SL_2(\mathbb{C})$ -representations as discussed in [B-H]. For a more general theory see [L-M].

For any finitely generated group G one can assign a commutative *universal representation* \mathbb{C} -algebra, $A[G]$, and a *universal representation*

$$\rho_G : G \rightarrow SL_2(A[G]),$$

with the following universal property:

For any \mathbb{C} -algebra A and any representation $\rho : G \rightarrow SL_2(A)$ there is a unique homomorphism of algebras $h_\rho : A[G] \rightarrow A$ which induces a homomorphism of groups

$$SL_2(h_\rho) : SL_2(A[G]) \rightarrow SL_2(A)$$

such that the following diagram commutes:

$$\begin{array}{ccc}
G & \xrightarrow{\rho_G} & SL_2(A[G]) \\
& \searrow \rho & \downarrow SL_2(h_\rho) \\
& & SL_2(A)
\end{array}$$

This universal property uniquely determines $A[G]$ up to an isomorphism. One can think of $A[G]$ as an (unreduced) coordinate ring of a scheme whose points, that is homomorphisms $A[G] \rightarrow \mathbb{C}$, are in a natural bijection with $SL_2(\mathbb{C})$ -representations of G . $GL_2(\mathbb{C})$ acts on these representations by conjugation. A standard argument from invariant theory states that there is a categorical quotient of this action, $A[G]^{GL_2(\mathbb{C})}$, called a *universal character ring* of G . Its \mathbb{C} -points represent semisimple $SL_2(\mathbb{C})$ -representations of G , or equivalently $SL_2(\mathbb{C})$ -characters of G . $A[G]^{GL_2(\mathbb{C})}$ is the most important object in the study of SL_2 -representations of G since it encodes all information necessary for their understanding (or classification). One of the most important results of [B-H], Proposition 9.1 (as well as $A^*.9.11$), gives a simple algebraic description of $A[G]^{GL_2(\mathbb{C})}$ by showing that $A[G]^{GL_2(\mathbb{C})}$ is isomorphic to the algebra $TH_{\mathbb{C}}(G)$ introduced in Section 3. The isomorphism carries $[g] + [g^{-1}] \in TH_{\mathbb{C}}(G) \subset H_{\mathbb{C}}(G)$, for $g \in G$, to $Tr(\rho_G(g))$. This result and Theorem 3.2 imply the following theorem.

Theorem 7.1

There is an isomorphism of \mathbb{C} -algebras $\phi : \mathcal{S}(G; R) \rightarrow A[G]^{GL_2(\mathbb{C})}$ such that $\phi([g]) = Tr(\rho_G(g))$, for any $g \in G$.

The above theorem implies in particular the following corollary.

Corollary 7.2

1. *Let G be any finitely generated group and $\chi : G \rightarrow \mathbb{C}$ be an $SL_2(\mathbb{C})$ -character of G , that is, the trace of an $SL_2(\mathbb{C})$ -representation of G . Then there is a unique homomorphism of \mathbb{C} -algebras $h_\chi : \mathcal{S}(G; \mathbb{C}) \rightarrow \mathbb{C}$ such that $h_\chi([g]) = \chi(g)$. If χ and χ' are two different $SL_2(\mathbb{C})$ -characters then $h_\chi \neq h_{\chi'}$.*
2. *Conversely, every homomorphism $h : \mathcal{S}(G; R) \rightarrow \mathbb{C}$ is induced by an $SL_2(\mathbb{C})$ -character of G , i.e. $h = h_\chi$, for some $\chi : G \rightarrow \mathbb{C}$.*

Remarks

1. Doug Bullock was the first to observe that the well-known equality (see e.g [Vo], [F-K], [Ho])

$$\chi(a)\chi(b) = \chi(ab) + \chi(ab^{-1}),$$

satisfied for any $SL_2(\mathbb{C})$ -character χ and $g, h \in G$ is very similar to the skein relation

$$[L_+] = A[L_0] + A^{-1}[L_\infty] \text{ in } \mathcal{S}_{2,\infty}(M; R, A) \text{ for } A = -1.$$

As an immediate consequence of this observation he got Corollary 7.2(1); compare [B-1], [B-2], [P-S-1]. This is the easier part of Corollary 7.2. He also proved independently from us Corollary 7.2(2), [B-4].

2. The statement of Theorem 7.1 is much stronger than Corollary 7.2. For example $A[G]^{GL_2(\mathbb{C})}$ may have nilpotent elements, which obviously will be undetected by homomorphisms $A[G]^{GL_2(\mathbb{C})} \rightarrow \mathbb{C}$. For more information on $A[G]^{GL_2(\mathbb{C})}$, and a discussion of possible nilpotent elements in this ring see [B-H]. We will come back to the question of nilpotents later in this section.
3. The second author proved (see [Si]) that Corollary 7.2 is true not only for the field of complex numbers but for all algebraically closed fields of characteristic $\neq 2$. Moreover, it can be generalized to fields which are not closed and to Dedekind domains.
4. The connections between skein modules and character varieties can be generalized to other algebraic groups. In particular, an analogous description of the universal $SL_n(\mathbb{C})$ -character ring of $\pi_1(M)$ in terms of links in M is given in [S-F].

Corollary 7.2 can be formulated in a different way. Let $X(G)$ denote the set of all $SL_2(\mathbb{C})$ -characters of G . For any $g \in G$ there is a function $\tau_g : X(G) \rightarrow \mathbb{C}$ given by $\tau_g(\chi) = \chi(g)$. It is known that $X(G)$ has a natural structure of an algebraic set determined by the requirement that each τ_g is a regular function on $X(G)$; see for example [C-S], [Bau]. The algebraic set $X(G)$ is called *the $SL_2(\mathbb{C})$ -character variety of G* . Its coordinate ring,

$\mathbb{C}[X(G)]$, is isomorphic to $(A[G]^{GL_2(\mathbb{C})})/\sqrt{0}$, and therefore to $\mathcal{S}(G; \mathbb{C})/\sqrt{0}$, where $\sqrt{0}$ denotes the radical of the ring, that is the ideal composed of all nilpotent elements. The isomorphism between $\mathcal{S}(G; \mathbb{C})/\sqrt{0}$ and $\mathbb{C}[X(G)]$ carries $[g]$ to τ_g . In this context it is natural to ask when $\mathcal{S}(G; \mathbb{C})$ has no nilpotent elements and therefore $\mathcal{S}(G; \mathbb{C}) \simeq \mathbb{C}[X(G)]$. The next theorem gives a list of all classes of groups for which we know a positive answer to this question.

Theorem 7.3

If one of the following conditions is satisfied

1. *G is finite*
2. *G is abelian*
3. *G is free non-abelian*
4. *G is the fundamental group of a surface satisfying condition (1) or (2) of Theorem 4.7*
5. *G is the knot group of a 2-bridge knot*

then $\mathcal{S}(G; \mathbb{C})$ does not have nilpotent elements and, therefore, $\mathcal{S}(G; \mathbb{C}) \simeq \mathbb{C}[X(G)]$.

Proof:

(1) If G is finite then by Maschke's Theorem $\mathbb{C}G$ is a semisimple algebra. By Wedderburn-Artin theory (see e.g. [C-R],[Hu]) any homomorphic image of a semisimple, artinian algebra is also semisimple and artinian. Therefore $\mathbb{C}G/I$, where I is generated by $g(h+h^{-1})-(h+h^{-1})g$, $g, h \in G$, is semisimple and artinian. Hence, by Wedderburn-Artin Theorem there exists an isomorphism $\phi : \mathbb{C}G/I \rightarrow M_{n_1}(\mathbb{C}) \times \dots \times M_{n_k}(\mathbb{C})$, where $M_{n_i}(\mathbb{C})$ is the algebra of $n_i \times n_i$ matrices with complex coefficients. Since $TH_{\mathbb{C}}(G)$ is generated by the elements $g + g^{-1} + I \in \mathbb{C}G/I$ and these elements are in the center of $\mathbb{C}G/I$, $\phi(TH_{\mathbb{C}}(G)) \subset \text{Center}(M_{n_1}(\mathbb{C}) \times \dots \times M_{n_k}(\mathbb{C})) = \mathbb{C}^k$. Therefore $\phi(TH_{\mathbb{C}}(G))$ has no nilpotent elements and, finally, $TH_{\mathbb{C}}(G)$ is nilpotent free, because ϕ is an isomorphism.

(2) It has been proved in [P-S-1] (Theorem 3.3).

(4) It follows immediately from Theorem 4.7

(3) This is a special case of (4).

(5) The Burde's and Zieschang's book [B-Z] is a good source of information about 2-bridge knots. Our proof will base on results of L. Thang [Th] on $Sl_2(\mathbb{C})$ -representations of 2-bridge knots ⁷. Let K be a 2-bridge knot. Then its group $G = \pi_1(S^3 \setminus K)$ has a presentation $G = \langle a, b | wa = bw \rangle$, where $w = a^{\epsilon_1} b^{\epsilon_n} a^{\epsilon_2} b^{\epsilon_{n-1}} \dots a^{\epsilon_n} b^{\epsilon_1}$, $\epsilon_i = \pm 1$. Let $F_2 = \langle a, b \rangle$. By Fact 2.6(3) the natural epimorphism $\pi : F_2 \rightarrow G$ yields an epimorphism $\pi_* : \mathcal{S}(F_2; \mathbb{C}) \rightarrow \mathcal{S}(G; \mathbb{C})$. Let $t_1 = [a]$, $t_2 = [ab]$, $t_3 = [b]$, $t_1, t_2, t_3 \in \mathcal{S}(F_2; \mathbb{C})$. Then by Corollary 4.5 $\mathcal{S}(F_2; \mathbb{C}) = \mathbb{C}[t_1, t_2, t_3]$. Since $[w], [bwa^{-1}] \in \mathcal{S}(F_2, \mathbb{C})$, $[w] = P_w(t_1, t_2, t_3)$, $[bwa^{-1}] = P_{bwa^{-1}}(t_1, t_2, t_3)$, where $P_w, P_{bwa^{-1}}$ are polynomials in variables t_1, t_2, t_3 . Then

$$P_w - P_{bwa^{-1}} \in \text{Ker} \pi_* \triangleleft \mathbb{C}[t_1, t_2, t_3] \quad (1)$$

Moreover, a and b are conjugate in G . Therefore $[a] = [b]$ in $\mathcal{S}(G; \mathbb{C})$ and

$$t_1 - t_3 \in \text{Ker} \pi_*. \quad (2)$$

Since $[a]$ and $[ab]$ generate $\mathcal{S}(G; \mathbb{C})$, a and ab distinguish all $Sl_2(\mathbb{C})$ -characters of G . Hence, by identifying any $\chi \in X(G)$ with a pair $(t_1, t_2) \in \mathbb{C}^2$, where $t_1 = \chi(a)$, $t_2 = \chi(ab)$, we can consider $X(G)$ as a subset of \mathbb{C}^2 . Le Thang has shown in [Th] (Theorem 3.3.1) that $X(G)$ is the zero set of a polynomial equation $P_w(t_1, t_2, t_1) - P_{bwa^{-1}}(t_1, t_2, t_1) = 0$. Moreover, he has shown (see [Th] Proposition 3.4.1) that $P_w(t_1, t_2, t_1) - P_{bwa^{-1}}(t_1, t_2, t_1) = (t_1^2 - t_2 - 2)\Phi(t_1, t_2)$, where $\Phi \in \mathbb{C}[t_1, t_2]$ has no multiple divisors. He also proved ([Th] Lemma 3.3.6) that $\Phi(2, 2) \neq 0$. Since $t_1^2 - t_2 - 2 = 0$ for $t_1 = t_2 = 2$, $t_1^2 - t_2 - 2$ does not divide $\Phi(t_1, t_2)$. Moreover, $t_1^2 - t_2 - 2$ is an irreducible polynomial. Therefore $P_w(t_1, t_2, t_1) - P_{bwa^{-1}}(t_1, t_2, t_1)$ has no quadratic divisors. This implies that $(P_w(t_1, t_2, t_1) - P_{bwa^{-1}}(t_1, t_2, t_1)) \triangleleft \mathbb{C}[t_1, t_2]$ is a radical ideal and

$$\mathbb{C}[X(G)] = \mathbb{C}[t_1, t_2] / (P_w(t_1, t_2, t_1) - P_{bwa^{-1}}(t_1, t_2, t_1)) \quad (3)$$

From the discussion preceding Theorem 7.3 follows that there is a homomorphism

$$\phi : \mathcal{S}(G; \mathbb{C}) \rightarrow \mathbb{C}[X(G)] = \mathbb{C}[t_1, t_2] / (P_w(t_1, t_2, t_1) - P_{bwa^{-1}}(t_1, t_2, t_1))$$

⁷These representations were also studied in [Ri] and [B-H].

such that $\phi([a]) = t_1, \phi([ab]) = t_2$. Therefore

$$\phi \circ \pi_* : \mathbb{C}[t_1, t_2, t_3] \rightarrow \mathcal{S}(G; \mathbb{C}) \rightarrow \mathbb{C}[X(G)] = \mathbb{C}[t_1, t_2] / (P_w(t_1, t_2, t_1) - P_{bwa^{-1}}(t_1, t_2, t_1))$$

maps t_1, t_3 on t_1 and t_2 on t_2 . Hence $\text{Ker} \pi_* \subset (t_1 - t_3, P_w(t_1, t_2, t_1) - P_{bwa^{-1}}(t_1, t_2, t_1)) = (t_1 - t_3, P_w(t_1, t_2, t_3) - P_{bwa^{-1}}(t_1, t_2, t_3)) \triangleleft \mathbb{C}[t_1, t_2, t_3]$. But we have observed in (1) and (2) that $t_1 - t_3 \in \text{Ker} \pi_*$ and $P_w(t_1, t_2, t_3) - P_{bwa^{-1}}(t_1, t_2, t_3) \in \text{Ker} \pi_*$. Therefore $\text{Ker} \pi_* = (P_w(t_1, t_2, t_1) - P_{bwa^{-1}}(t_1, t_2, t_1))$. Hence $\mathcal{S}(G; \mathbb{C}) \simeq \mathbb{C}[t_1, t_2, t_3] / \text{Ker} \pi_* \simeq \mathbb{C}[X(G)]$ has no nilpotent elements. \square

Of course Theorem 4.7 implies much more than the statement of Theorem 7.3(4).

Corollary 7.4

Suppose that F is either an orientable surface or F is an unorientable surface of an even, negative Euler characteristic. Then the $Sl_2(\mathbb{C})$ -character variety $X(\pi_1(F))$ is an irreducible affine algebraic set.

$Sl_2(\mathbb{C})$ -character varieties of surfaces were investigated earlier by several authors. In particular, W. Goldman constructed a Poisson bracket on coordinate rings of $Sl_2(\mathbb{C})$ -character varieties of closed surfaces (See [Go]). D. Bullock, C. Frohman and J. Kania-Bartoszyńska showed that the skein algebra $\mathcal{S}_{2,\infty}(F \times I; \mathbb{C}[A^{\pm 1}], A)$ can be considered as a quantization of $\mathbb{C}[X(\pi_1(F))]$ according to the Goldman-Poisson bracket, [B-F-K].

Using our Theorem 7.1 one can reformulate some of the results of Culler and Shalen stated in [C-S]. In particular, they proved that if M is a compact, oriented 3-manifold and an irreducible component of the $Sl_2(\mathbb{C})$ -character variety of $\pi_1(M)$ has dimension greater or equal to 1 then M has an incompressible surface not parallel to the boundary of M . Bullock observed that the result of Culler and Shalen combined with the statement of Theorem 7.1 implies the following corollary.

Corollary 7.5

If the skein module, $\mathcal{S}_{2,\infty}(M; \mathbb{C}, -1)$, of a compact, oriented 3-manifold M is infinite dimensional (as a \mathbb{C} -linear space) then M has an incompressible surface which is not boundary-parallel.

The above corollary gives a topological condition on links in M (modulo skein relations) which implies the existence an incompressible surface in M .

The only existing proof of Corollary 7.5 is very complicated. It would be very desirable to find a new, simpler and purely topological proof of this corollary.

Corollary 7.5 shows that $\mathcal{S}_{2,\infty}(M; \mathbb{Z}[A^{\pm 1}], A)$ carries a great amount of important information about the manifold M .⁸

8 Estimating minimal numbers of generators of skein algebras, Part 2

In this section we are going to prove theorems similar to theorems of Section 6, but concerning the case in which $A^2 + A^{-2}$ is invertible in rings of coefficients of skein algebras. We use the notation introduced in Section 6.

Theorem 8.1

Let A be an invertible element in a ring R such that $(A^2 + A^{-2})^{-1} \in R$. Then $\mathcal{S}_{2,\infty}(F_{g,n} \times I; R, A)$ is generated by $N + \binom{N}{2} + \binom{N}{3}$ elements $[K_S]$, where $N = \text{rank} H_1(F_{g,n})$ and $S \subset \{1, \dots, N\}, S \neq \emptyset, \overline{\overline{S}} \leq 3$.⁹

Proof: Since any link in $F_{g,0}$ can be pushed into $F_{g,1} \subset F_{g,0}$, the embedding $i : F_{g,1} \rightarrow F_{g,0}$ induces an epimorphism $i_* : \mathcal{S}_{2,\infty}(F_{g,1} \times I; R, A) \rightarrow \mathcal{S}_{2,\infty}(F_{g,0} \times I; R, A)$. Hence, it is enough to prove Theorem 8.1 only for surfaces $F_{g,n}$, $n \geq 1$.

Let us consider a homomorphism $r : \mathbb{Z}[A^{\pm 1}] \rightarrow R$, $r(A) = A \in R$. By Fact 2.7 there is an isomorphism

$$\bar{r} : \mathcal{S}_{2,\infty}(F_{g,n} \times I; R, A) \rightarrow \mathcal{S}_{2,\infty}(F_{g,n} \times I; \mathbb{Z}[A^{\pm 1}], A) \otimes_{\mathbb{Z}[A^{\pm 1}]} R.$$

Therefore Theorem 6.1 implies that $\mathcal{S}_{2,\infty}(F_{g,n} \times I; R, A)$ is generated by the elements $[K_S]$, $S \subset \{1, \dots, N\}$. Hence it is enough to prove that the subalgebra $P \subset \mathcal{S}_{2,\infty}(F_{g,n} \times I; R, A)$ generated by $[K_S]$, $\overline{\overline{S}} \leq 3$, contains all elements $[K_S]$, $S \subset \{1, 2, \dots, N\}$, $S \neq \emptyset$.

Suppose that $S_0 = \{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, N\}$, $i_1 < i_2 < \dots < i_k$, $k \geq 4$, is a set of the smallest cardinality such that $[K_{S_0}] \notin P$. Let us consider knots $K_{\{i_1, i_3\}}$ and $K_{S_0 \setminus \{i_1, i_3\}}$. Then, depending on S_0 , the minimal number of intersections between $K_{\{i_1, i_3\}}$ and $K_{S_0 \setminus \{i_1, i_3\}}$ is equal to 2, 3 or 4. Let us consider each of these possibilities.

⁸ Notice that $\mathcal{S}_{2,\infty}(M; \mathbb{C}, -1) = \mathcal{S}_{2,\infty}(M; \mathbb{Z}[A^{\pm 1}], A) \otimes_{\mathbb{Z}[A^{\pm 1}]} \mathbb{C}$.

⁹ $\overline{\overline{S}}$ denotes the number of elements of S .

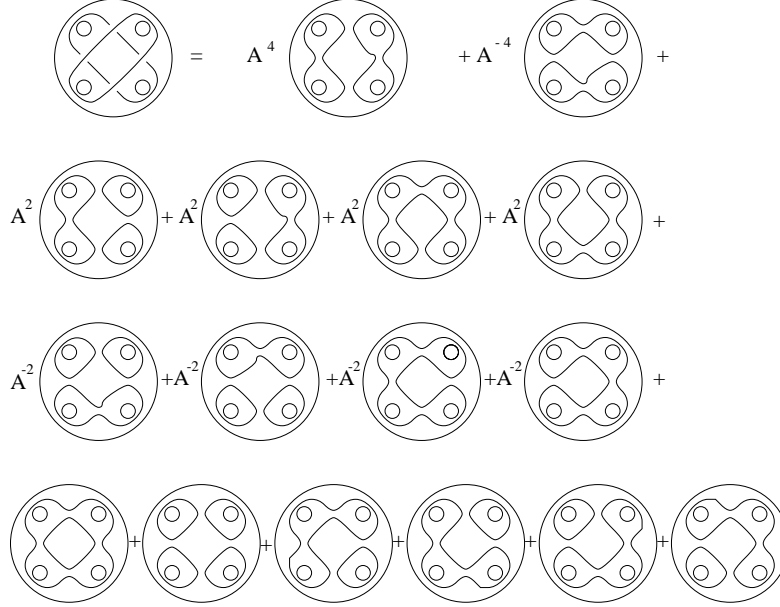


Figure 7

1. The minimal number of intersections between $K_{\{i_1, i_3\}}$ and $K_{S_0 \setminus \{i_1, i_3\}}$ is four.

One can check by straightforward, but long computations that

$$\begin{aligned}
[K_{\{i_1, i_3\}}][K_{S_0 \setminus \{i_1, i_3\}}] &= (A^2 + A^{-2})[K_{S_0}] + A^4[K_{\{i_1, i_2\}}][K_{S_0 \setminus \{i_1, i_2\}}] + \\
&A^{-4}[K_{\{i_2, i_3\}}][K_{S_0 \setminus \{i_2, i_3\}}] + A^2[K_{\{i_1, i_2\}}][K_{\{i_3\}}][K_{S_0 \setminus \{i_1, i_2, i_3\}}] + \\
&A^2[K_{\{i_1\}}][K_{\{i_2\}}][K_{S_0 \setminus \{i_1, i_2\}}] + A^{-2}[K_{\{i_2\}}][K_{\{i_3\}}][K_{S_0 \setminus \{i_2, i_3\}}] + \\
&A^{-2}[K_{\{i_2, i_3\}}][K_{\{i_1\}}][K_{S_0 \setminus \{i_1, i_2, i_3\}}] + [K_{\{i_1\}}][K_{\{i_2\}}][K_{\{i_3\}}][K_{S_0 \setminus \{i_1, i_2, i_3\}}] + \\
&[K_{\{i_2\}}][K_{S_0 \setminus \{i_2\}}] + [K_{\{i_3\}}][K_{S_0 \setminus \{i_3\}}] + \\
&[K_{\{i_1, i_2, i_3\}}][K_{S_0 \setminus \{i_1, i_2, i_3\}}] + [K_{\{i_1\}}][K_{S_0 \setminus \{i_1\}}].
\end{aligned}$$

The above calculations for the knot K_{S_0} placed in $F_{0,5}$, $S_0 = \{1, 2, 3, 4\}$, are shown in Fig. 7.

Since all terms in the above equation, except $(A^2 + A^{-2})[K_{S_0}]$ involve only elements $[K_S]$, for $\overline{S} < \overline{S_0}$, we get $(A^2 + A^{-2})[K_{S_0}] \in P$. Therefore $[K_{S_0}] \in P$, what contradicts our earlier assumption.

2. The minimal number of intersections between $K_{\{i_1, i_3\}}$ and $K_{S_0 \setminus \{i_1, i_3\}}$ is three.

Notice that then $K_{\{i_1, i_3\}}, K_{S_0 \setminus \{i_1, i_3\}} \subset F_{g,n}$, where $g \geq 1, n \geq 3$. We

will assume, for simplicity, that $g = 1, n = 3$. The proof for all other surfaces $F_{g,n}$ is identical. The assumptions $g = 1, n = 3$ imply that $S_0 = \{1, 2, 3, 4\}$. The product of $K_{\{i_1, i_3\}}$ and $K_{S_0 \setminus \{i_1, i_3\}}$ is shown in Fig. 8.

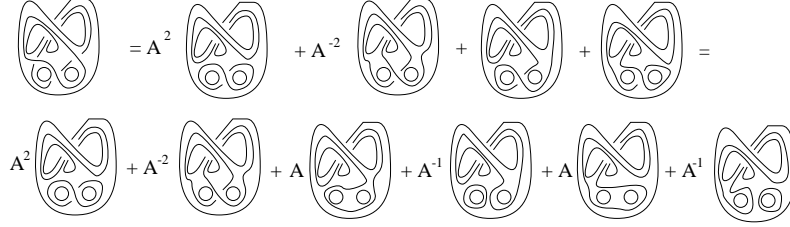




Figure 8

Notice that any knot of the form  considered as an element of a skein module is equal to $A^{-1} \langle \text{diagram} \rangle - A^{-2} \langle \text{diagram} \rangle$. Therefore $\langle \text{diagram} \rangle, \langle \text{diagram} \rangle \in P$. Similarly $\langle \text{diagram} \rangle \in P$. Moreover, $\langle \text{diagram} \rangle, \langle \text{diagram} \rangle \in P$. Hence $A \langle \text{diagram} \rangle + A \langle \text{diagram} \rangle \in P$. But $A \langle \text{diagram} \rangle + A \langle \text{diagram} \rangle = A^2 \langle \text{diagram} \rangle - A^3 \langle \text{diagram} \rangle + \langle \text{diagram} \rangle - A^{-1} \langle \text{diagram} \rangle$. Therefore, $(A^3 + A^{-1}) \langle \text{diagram} \rangle \in P$. Since $A^3 + A^{-1} = A(A^2 + A^{-2})$ is invertible in R we get a contradiction.

3. The minimal number of intersections between $K_{\{i_1, i_3\}}$ and $K_{S_0 \setminus \{i_1, i_3\}}$ is two.

Notice that then $g \geq 2$. In this part of the proof we will assume, for simplicity, that $g = 2$ and $n = 1$. (For all different surfaces the proof is identical). Then $S_0 = \{1, 2, 3, 4\}$ and $[K_{S_0}] = \langle \text{diagram} \rangle$. Observe that

$$\langle \text{diagram} \rangle = A \cdot \langle \text{diagram} \rangle + A^{-1} \langle \text{diagram} \rangle. \text{ Since } \langle \text{diagram} \rangle, \langle \text{diagram} \rangle \in P, \text{ we get}$$

 $\in P$. Hence

$$\text{Diagram} = A^{\epsilon_1} \cdot \text{Diagram}_1 + A^{-\epsilon_1} \cdot \text{Diagram}_2 \in P, \quad (4)$$

where $\epsilon_1 = +1$ or $\epsilon_1 = -1$ depending on the type of the crossing in . Moreover,

$$\text{Diagram} = A^{\epsilon_2} \cdot \text{Diagram}_3 + A^{-\epsilon_2} \cdot \text{Diagram}_4 \in P. \quad (5)$$

After combining equations (4) and (5) we get

$$\text{Diagram} - A^{-2\epsilon_1} A^{-2\epsilon_2} \text{Diagram}_5 \in P. \quad (6)$$

Similarly, we have

$$\text{Diagram} - A^{-2\epsilon'_1} A^{-2\epsilon'_2} \text{Diagram}_6 \in P. \quad (7)$$

Finally we use the following equation:

$$\text{Diagram} = A^2 \cdot \text{Diagram}_7 + A^{-2} \cdot \text{Diagram}_8 + \text{Diagram}_9 + \text{Diagram}_{10}. \quad (8)$$

Since the first three links in the above equation belong to P we see that

$\text{Diagram}_9 + \text{Diagram}_{10} \in P$. Therefore, after combining the above equation

with equations (6) and (7) we get $(A^{-2\epsilon_1-2\epsilon_2} + A^{-2\epsilon'_1-2\epsilon'_2}) \cdot \text{Diagram}_5 \in P$,

where $\epsilon_1, \epsilon_2, \epsilon'_1, \epsilon'_2 \in \{+1, -1\}$. In particular, we get $(A^4 + 1) \cdot \text{Diagram}_5 \in P$.

But $A^4 + 1 = (A^2 + A^{-2})A^2$ is invertible in R . Hence $[K_{S_0}] =$

$\text{Diagram}_5 \in P$.

This completes the proof of Theorem 8.1.

□

Remark 8.2

One can observe that sometimes it is possible to reduce the number of generators of $\mathcal{S}_{2,\infty}(F_{g,n} \times I; R, A)$ even if $A^2 + A^{-2}$ is not invertible in R . In particular, it is always possible to do this if at least one element of the ideal $(2, A^2 + A^{-2}) \triangleleft R$ is invertible in R (i.e. $(2, A^2 + A^{-2}) = R$) and $g \geq 1, n \geq 2$.

Theorems 6.1 and 8.1 may be useful for estimating Heegaard genera of 3-manifolds. Notice that if $H_1 \cup H_2 = M$ is a Heegaard splitting of M then the embedding $i : H_1 \rightarrow M$ induces an epimorphism of R -algebras $\mathcal{S}_{2,\infty}(H_1; R, -1) \rightarrow \mathcal{S}_{2,\infty}(M; R, A)$. Therefore Theorem 6.1 and Theorem 8.1 imply the following fact.

Proposition 8.3

Let $g(M)$ be the Heegaard genus of a compact 3-manifold M . Let us denote the minimal number of generators of any R -algebra A by $\text{rank}(A)$. Then

1. $g(M) \geq \log_2 \left(\text{rank}(\mathcal{S}_{2,\infty}(M; R, -1)) + 1 \right)$
2. $g(M) + \binom{g(M)}{2} + \binom{g(M)}{3} \geq \text{rank}(\mathcal{S}_{2,\infty}(M; R, -1))$ if $\frac{1}{2} \in R$.

The above inequalities are very closely related to the obvious inequality

$$g(M) \geq \text{rank}(\pi_1(M)), \quad (9)$$

where $\text{rank}(G)$ denotes the minimal number of generators of G . However, the inequalities of Proposition 8.3 have some advantages over (9). Notice that $\text{rank}(\mathcal{S}_{2,\infty}(M; \mathbb{C}, -1)) \geq \text{rank}(\mathbb{C}[X(G)])$, where $G = \pi_1(M)$. In practice, the minimal number of generators of $\mathbb{C}[X(G)]$ can be estimated by using methods of computational algebraic geometry. For example, one can calculate dimensions of tangent spaces of some irreducible components of $X(G)$ at singular points. The minimal number of generators of $\mathbb{C}[X(G)]$ (and hence, of $\mathcal{S}_{2,\infty}(M; R, -1)$) cannot be lower than any of these dimensions.

We will use this method in the latter part of this section.

Theorem 8.4

Let G be a group generated by $g_1, g_2, \dots, g_n \in G$ and let R be a ring such that $1/2 \in R$. Then

1. $\mathcal{S}(G; R)$ can be generated by $n + \binom{n}{2} + \binom{n}{3}$ elements of the form $[g_i], [g_j g_k], [g_s g_t g_v]$, for $1 \leq i \leq n, 1 \leq j < k \leq n, 1 \leq s < t < v \leq n$.
2. If G is abelian then $\mathcal{S}(G; R)$ can be generated by $n + \binom{n}{2}$ elements $[g_i], [g_j g_k], 1 \leq i \leq n, 1 \leq j < k \leq n$.

Proof:

1. Let $\pi : F_n = \langle a_1, \dots, a_n \rangle \rightarrow G$ be an epimorphism, $\pi(a_i) = g_i, i = 1, 2, \dots, n$. By Fact 2.6(3) π induces an epimorphism $\pi_* : \mathcal{S}(F_n; R) \rightarrow \mathcal{S}(G; R)$ $\pi_*([a_i]) = [g_i]$. Therefore it is enough to show that the elements $[a_i], [a_j a_k], [a_s a_t a_v]$, for $1 \leq i \leq n, 1 \leq j < k \leq n, 1 \leq s < t < v \leq n$, generate $\mathcal{S}(F_n; R)$.

Since $\frac{1}{2} \in R$, Theorem 8.1 implies that $\mathcal{S}_{2,\infty}(F_{0,n+1} \times I; R, -1)$ is generated by elements $[K_{\{i\}}], [K_{\{j,k\}}], [K_{\{s,t,v\}}]$. By Theorem 2.8

$$\hat{\xi} : \mathcal{S}_{2,\infty}(F_{0,n+1} \times I; R, -1) \rightarrow \mathcal{S}(\langle a_1, a_2, \dots, a_n \rangle; R)$$

is an isomorphism of algebras such that $\hat{\xi}([K_{\{i\}}]) = -[a_i]$. Therefore $\mathcal{S}(F_n; R)$ is generated by elements $[a_i], [a_j a_k], [a_s a_t a_v]$, for $1 \leq i \leq n, 1 \leq j < k \leq n, 1 \leq s < t < v \leq n$,

2. Let \mathbb{Z}^n be a free abelian group generated by g'_1, g'_2, \dots, g'_n . Let $\pi : \mathbb{Z}^n \rightarrow G$ be an epimorphism, $\pi(g'_i) = g_i$. As in the proof of (1) it is enough to show that $\mathcal{S}(\mathbb{Z}^n; R)$ can be generated by $g'_i, g'_j g'_k$, for $1 \leq i \leq n, 1 \leq j < k \leq n$.

Let us consider the group ring $R\mathbb{Z}^n \simeq R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$. Let $R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]^{sym}$ denote a subring of $R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ consisted of constant elements under an involution $\tau : R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}] \rightarrow R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}], \tau(x_i) = x_i^{-1}$, for $i = 1, 2, \dots, n$.

We have proved in [P-S-1] (Theorem 2.3) that there exists an isomorphism $\hat{\phi} : \mathcal{S}(G; R) \rightarrow R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]^{sym}$ such that $\hat{\phi}([g_i]) = x_i + x_i^{-1}$. Therefore the proof of Theorem 8.4(2) will be completed if we show the following lemma.

Lemma 8.5

The ring $R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]^{sym}$ is generated by elements $x_i + x_i^{-1}, x_j x_k + x_j^{-1} x_k^{-1}, 1 \leq i \leq n, 1 \leq j < k \leq n$.

Proof: Let $a_i = x_i + x_i^{-1}, b_i = x_i - x_i^{-1}$, for $i = 1, 2, \dots, n$. Since $1/2 \in R$, $x_i = \frac{a_i + b_i}{2}, y_i = \frac{a_i - b_i}{2}$ and the ring $R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ is generated by $a_1, \dots, a_n, b_1, \dots, b_n$. Every element $v \in R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]^{sym}$ is a sum $v = \sum m_i$ such that every m_i is a monomial of the form $m_i = ca_1^{\alpha_1} \dots a_n^{\alpha_n} b_1^{\beta_1} \dots b_n^{\beta_n}$, where $c \in R, \alpha_i, \beta_i \in \{0, 1, 2, \dots\}$. Then $v = \frac{v + \tau(v)}{2} = \sum \frac{m_i + \tau(m_i)}{2}$. Notice that $\tau(a_i) = a_i, \tau(b_i) = -b_i$. Hence

$$\frac{m_i + \tau(m_i)}{2} = \begin{cases} m_i & \text{if } \sum_{j=1}^n \beta_j \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

Therefore v is a sum of monomials of the form $ca_1^{\alpha_1} \dots a_n^{\alpha_n} b_1^{\beta_1} \dots b_n^{\beta_n}$, where $\sum_{j=1}^n \beta_j$ is even. Hence $R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]^{sym}$ is generated by elements $a_i, b_j b_k, 1 \leq i \leq n, 1 \leq j < k \leq n$. But $a_j a_k + b_j b_k = 2(x_j x_k + x_j^{-1} x_k^{-1})$. Therefore $R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]^{sym}$ is generated by $a_i = x_i + x_i^{-1}$ and $x_j x_k + x_j^{-1} x_k^{-1}$, for $1 \leq i \leq n, 1 \leq j < k \leq n$. \square

Our next goal is to prove a theorem analogous to Theorem 6.3(3) but concerning rings R such that $A^2 + A^{-2}$ is invertible in R .

Theorem 8.6

Let a ring R be an integral domain of characteristic 0. Let G be a free abelian or nonabelian group on n generators. Then the numbers of generators given in Theorem 8.4 are minimal i.e.

1. $\mathcal{S}(F_n; R)$ cannot have fewer than $n + \binom{n}{2} + \binom{n}{3}$ generators.
2. $\mathcal{S}(\mathbb{Z}^n; R)$ cannot have fewer than $n + \binom{n}{2}$ generators.

We are going to prove Theorem 8.6 using methods of algebraic geometry. Let G be any group and let $\chi_0 : G \rightarrow \mathbb{C}$ be the trace of the trivial $Sl_2(\mathbb{C})$ -representation (i.e. $\chi_0 = 2$). For a proof of Theorem 8.6 we will need to calculate dimensions of tangent spaces $T_{\chi_0} X(G)$, for $G = F_n, \mathbb{Z}^n$. These dimensions will estimate from below the minimal numbers of generators of the corresponding skein algebras.

Lustig and Metzler showed in [L-M] (Theorem 1) the following result.

Fact 8.7

The tangent space $T_{\chi_0} X(F_n)$ has dimension $n + \binom{n}{2} + \binom{n}{3}$.

Using Lustig's and Metzler's idea and the lemma below we will show that $\dim T_{\chi_0} X(\mathbb{Z}^n) = n + \binom{n}{2}$.

Lemma 8.8

Let $e_1 = (1, 0), e_2 = (0, 1), e_1, e_2 \in \mathbb{Z}^2$. Then any $Sl_2(\mathbb{C})$ -character χ of \mathbb{Z}^2 is uniquely determined by the values $a_1 = \chi(e_1), a_2 = \chi(e_2)$ and $b = \chi(1, 1)$. Therefore $X(\mathbb{Z}^2) \subset \mathbb{C}^3$. Moreover, $X(\mathbb{Z}^2) = \{(a_1, a_2, b) \in \mathbb{C}^3 : a_1^2 + a_2^2 + b^2 - a_1 a_2 b - 4\}$.¹⁰

Proof: By Theorem 8.4(2) $\mathcal{S}(\mathbb{Z}^2; \mathbb{C})$ is generated by $(1, 0), (0, 1), (1, 1) \in \mathbb{Z}^2$ and therefore, by the remarks preceding Theorem 7.3, $\mathbb{C}[X(\mathbb{Z}^2)]$ is generated by $\tau_{(1,0)}, \tau_{(0,1)}, \tau_{(1,1)}$. Hence we can assume that $X(\mathbb{Z}^2) \subset \mathbb{C}^3$ and any character $\chi \in X(\mathbb{Z}^2)$ has coordinates $(a_1, a_2, b) \in \mathbb{C}^3$, where $a_1 = \chi(1, 0), a_2 = \chi(0, 1), b = \chi(1, 1)$. Let $P(x, y, z) = x^2 + y^2 + z^2 - xyz - 4$. We are going to show that for any character $\chi \in X(\mathbb{Z}^2)$ $P(a_1, a_2, b) = 0$.

Let χ be the trace of a representation $\rho : \mathbb{Z}^2 \rightarrow Sl_2(\mathbb{C})$. Since we can replace ρ by any of its conjugate representations we can assume that $\rho(1, 0)$ has an upper triangular form, $\rho(1, 0) = \begin{pmatrix} c_1 & d_1 \\ 0 & c_1^{-1} \end{pmatrix}$. Since $\rho(0, 1)$ commutes with $\rho(1, 0)$, the matrix $\rho(0, 1)$ has also an upper triangular form, $\rho(0, 1) = \begin{pmatrix} c_2 & d_2 \\ 0 & c_2^{-1} \end{pmatrix}$. Then $\chi = tr \circ \rho'$, where ρ' is a diagonal representation

$$\rho'(1, 0) = \begin{pmatrix} c_1 & 0 \\ 0 & c_1^{-1} \end{pmatrix}, \quad \rho'(0, 1) = \begin{pmatrix} c_2 & 0 \\ 0 & c_2^{-1} \end{pmatrix}.$$

Hence $a_1 = c_1 + c_1^{-1}, a_2 = c_2 + c_2^{-1}, b = c_1 c_2 + c_1^{-1} c_2^{-1}$ and straightforward calculations show that $P(a_1, a_2, b) = 0$.

Suppose now that $P(a_1, a_2, b) = 0$ for some $(a_1, a_2, b) \in \mathbb{C}^3$. We are going to show that $a_1 = \chi(1, 0), a_2 = \chi(0, 1), b = \chi(1, 1)$ for some character $\chi \in X(\mathbb{Z}^2)$. Let $c_1, c_2 \in \mathbb{C}$ be such that $c_1 + c_1^{-1} = a_1, c_2 + c_2^{-1} = a_2$ and let $b' = c_1 c_2 + c_1^{-1} c_2^{-1}, b'' = c_1 c_2^{-1} + c_1^{-1} c_2$. There are three possible cases:

1. $c_1, c_2 \neq \pm 1$.

Since the points $(a_1, a_2, b'), (a_1, a_2, b'') \in \mathbb{C}^3$ correspond to the traces of

¹⁰Although this is a well-known fact, we had a trouble with finding a good reference for a proof. For that reason we decided to give an elementary proof here. We have seen earlier many times that many algebraic results concerning character varieties can be proven using topological methods (i.e. skein algebras). The description of the skein algebra $\mathcal{S}(S^1 \times S^1 \times I; R, A)$ given in [B-P] immediately gives a topological proof of Lemma 8.8.

representations $\rho, \rho' : \mathbb{Z}^2 \rightarrow Sl_2(\mathbb{C})$,

$$\rho(1, 0) = \begin{pmatrix} c_1 & 0 \\ 0 & c_1^{-1} \end{pmatrix}, \quad \rho(0, 1) = \begin{pmatrix} c_2 & 0 \\ 0 & c_2^{-1} \end{pmatrix}$$

$$\rho'(1, 0) = \begin{pmatrix} c_1 & 0 \\ 0 & c_1^{-1} \end{pmatrix}, \quad \rho'(0, 1) = \begin{pmatrix} c_2^{-1} & 0 \\ 0 & c_2 \end{pmatrix},$$

the first part of the proof implies that $P(a_1, a_2, b') = P(a_1, a_2, b'') = 0$. Moreover, $b' - b'' = (c_1 - c_1^{-1})(c_2 - c_2^{-1}) \neq 0$. Since the equation $P(a_1, a_2, z) = 0$ has at most 2 solutions, the equation $P(a_1, a_2, b) = 0$ implies that $b = b'$ or $b = b''$. Hence the trace of ρ or the trace of ρ' is a character of \mathbb{Z}^2 with the coordinates (a_1, a_2, b) .

2. $c_1 = 1$ or $c_2 = 1$.

Suppose that $c_2 = 1$ (for $c_1 = 1$ the proof is identical). Then $a_2 = 2$ and $a_1^2 + b^2 - 2a_1b = 0$. Therefore $a_1 = b$ and the trace of the representation $\rho : \mathbb{Z}^2 \rightarrow Sl_2(\mathbb{C})$,

$$\rho(1, 0) = \begin{pmatrix} c_1 & 0 \\ 0 & c_1^{-1} \end{pmatrix}, \quad \rho(0, 1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is the desired $Sl_2(\mathbb{C})$ -character of \mathbb{Z}^2 .

3. $c_1 = -1$ or $c_2 = -1$.

Suppose that $c_2 = -1$. Then $a_2 = -2, a_1 = -b$, and the trace of the representation $\rho : \mathbb{Z}^2 \rightarrow Sl_2(\mathbb{C})$,

$$\rho(1, 0) = \begin{pmatrix} c_1 & 0 \\ 0 & c_1^{-1} \end{pmatrix}, \quad \rho(0, 1) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the desired $Sl_2(\mathbb{C})$ -character of \mathbb{Z}^2 .

□

Theorem 8.9

$$\dim T_{X_0} X(\mathbb{Z}^n) = n + \binom{n}{2} = \binom{n+1}{2}.$$

Proof: Let $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_n = (0, 0, \dots, 1) \in \mathbb{Z}^n$. By Theorem 8.4(2) and the remarks preceding Theorem 7.3, $\mathbb{C}[X(\mathbb{Z}^n)]$ is generated by $\tau_{e_i}, \tau_{e_j + e_k}$, for $1 \leq i \leq n, 1 \leq j < k \leq n$. Therefore we assume

that $X(\mathbb{Z}^n) \subset \mathbb{C}^{n+\binom{n}{2}}$ and any $\chi \in X(\mathbb{Z}^n)$ has coordinates $c_i = \chi(e_i)$ and $d_{k,l} = \chi(e_k + e_l)$, $k < l$.

Let $h_{s,t} : \mathbb{Z}^n \rightarrow \mathbb{Z}^2$ be a homomorphism $h_{s,t}(x_1, \dots, x_n) = (x_s, x_t)$, for $s, t \in \{1, \dots, n\}$, $s < t$. Let $f_{s,t} : X(\mathbb{Z}^2) \rightarrow X(\mathbb{Z}^n)$ be a morphism such that $f_{s,t}(\chi) = \chi \circ h_{s,t}$. One can easily see that if $\chi \in X(\mathbb{Z}^2)$ has the coordinates $(a_1, a_2, b) \in \mathbb{C}^3$ (we use the notation from the previous lemma), then $f_{s,t}(\chi)$ has coordinates $c_i, d_{k,l}$, where $c_i = \chi \circ h_{s,t}(e_i)$, $d_{k,l} = \chi \circ h_{s,t}(e_k + e_l)$. Therefore

$$c_i = \begin{cases} a_1 & \text{if } i = s \\ a_2 & \text{if } i = t \\ 2 & \text{otherwise} \end{cases} \quad d_{k,l} = \begin{cases} a_1 & \text{if } s \in \{k, l\}, t \notin \{k, l\} \\ a_2 & \text{if } t \in \{k, l\}, s \notin \{k, l\} \\ b & \text{if } \{s, t\} = \{k, l\} \\ 2 & \text{otherwise} \end{cases}$$

Since all partial derivatives of $a_1^2 + a_2^2 + b^2 - a_1 a_2 b - 4$ disappear for $(a_1, a_2, b) = (2, 2, 2)$, the variety $X(\mathbb{Z}^2)$ has the tangent space at χ_0 isomorphic to \mathbb{C}^3 . Therefore the linear map $f_{s,t*} : T_{\chi_0} X(\mathbb{Z}^2) \rightarrow T_{\chi_0} X(\mathbb{Z}^n) \subset \mathbb{C}^{n+\binom{n}{2}}$ has a matrix presented in Figure 9. A star in an entry of the matrix denotes any number.

One can easily see that the subspaces $Im f_{s,t*} \subset \mathbb{C}^{n+\binom{n}{2}}$ for all $s, t \in \{1, \dots, n\}$, $s < t$ span the space $\mathbb{C}^{n+\binom{n}{2}}$. Since $X(\mathbb{Z}^n) \subset \mathbb{C}^{n+\binom{n}{2}}$, $dim T_{\chi_0} X(\mathbb{Z}^n) = n + \binom{n}{2}$. \square

Proof of Theorem 8.6:

Let $G = F_n$ and $N = n + \binom{n}{2} + \binom{n}{3}$ (respectively: $G = \mathbb{Z}^n$ and $N = n + \binom{n}{2}$). We denote the algebraic closure of the field of fractions of R by K . The assumptions about R in Theorem 8.6 imply that K has characteristic 0. We have noticed in the remark preceding Theorem 7.3 that there exists an epimorphism $\phi : \mathcal{S}(G; K) \rightarrow K[X(G)]$, where $X(G)$ denotes the $Sl_2(K)$ -character variety of G . Since the number of generators of $K[X(G)]$ cannot be smaller than the dimension of $T_{\chi_0} X(G)$, $K[X(G)]$, and therefore $\mathcal{S}(G; K)$, have at least N generators¹¹. By Fact 2.6(4) $\mathcal{S}(G; K) = \mathcal{S}(G; R) \otimes_R K$. Hence $\mathcal{S}(G; R)$ has at least N generators.

¹¹Here we use the fact that Lustig's and Metzler's theorem as well as our result on dimensions of tangent spaces of character varieties and all results of Section 7 are valid for any algebraically closed field of characteristic 0.

$$\begin{array}{c}
\begin{array}{c}
\mathbf{c}_1 \\
\vdots \\
\mathbf{c}_s \\
\vdots \\
\mathbf{c}_t \\
\vdots \\
\mathbf{c}_n \\
\mathbf{d}_{11} \\
\vdots \\
\mathbf{d}_{st} \\
\vdots \\
\mathbf{d}_{nn}
\end{array}
\begin{array}{ccc}
\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \\
\hline
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
0 & \vdots & \vdots \\
1 & \vdots & \vdots \\
0 & \vdots & \vdots \\
\vdots & 0 & \vdots \\
\vdots & 1 & \vdots \\
\vdots & 0 & \vdots \\
\vdots & \vdots & \vdots \\
0 & 0 & 0 \\
* & * & 0 \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & 0 \\
* & * & 1 \\
\vdots & \vdots & 0 \\
\vdots & \vdots & \vdots \\
* & * & 0
\end{array}
\end{array}$$

Figure 9: The matrix of $f_{s,t*}$.

9 Acknowledgments

The authors would like to express their gratitude to D. Bullock, G. Brumfiel, B. Goldman, C. Frohman and J. Millson for fruitful discussions. The second author would like to thank V. Jones for his support during the time in which this paper was written.

References

- [Bar] J. Barrett, Skein Spaces and Spin Structures, University of Nottingham (1995), preprint.
- [Bau] G. Baumslag, Topics in combinatorial group theory. Lectures in Mathematics ETH Zurich, Birkhauser Verlag, 1993.
- [B-H] G. W. Brumfiel, H. M. Hilden, $Sl(2)$ Representations of Finitely Presented Groups, *Contemp. Math.* **187** (1995).

- [B-1] D. Bullock, Estimating a Skein Module with $SL_2(\mathbb{C})$ characters, *Proc. Amer. Math. Soc.* **125** (1997), no. 6, 1835–1839.
- [B-2] D. Bullock, Estimating the States of the Kauffman Bracket Skein Module, to appear in Proceedings of The International Conference in Knot Theory, Warsaw 1995, *Banach Center Publications* **42**
- [B-3] D. Bullock, A finite set of generators for the Kauffman bracket skein algebra, preprint (1995).
- [B-4] D. Bullock, Rings of $SL_2(\mathbb{C})$ -characters and the Kauffman bracket skein module, preprint (1996).
- [B-5] D. Bullock, The $(2, \infty)$ -Skein Module of The Complement of a $(2, 2p + 1)$ Torus Knot, *J. Knot Theory Ramifications*, **4**(4) (1995), 619-632.
- [B-F-K] D. Bullock, C. Frohman, J. Kania-Bartoszyńska, Understanding The Kauffman Bracket Skein Module, to appear in *J. Knot Theory Ramifications*.
- [B-P] D. Bullock, J.H. Przytycki, Kauffman bracket skein module quantization of symmetric algebra and $so(3)$, to appear in *Proc. Amer. Math. Soc.*
- [B-Z] G. Burde and H. Zieschang, *Knots*, de Gruyter, (1985).
- [C-S] M. Culler and P.B. Shalen, Varieties of group representations and splittings of 3-manifolds, *Ann. of Math.* **117** (1983), 109-146.
- [C-R] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, J. Wiley, New York, London (1962).
- [F-K] R. Fricke, F. Klein, *Vorlesungen über die Theorie der automorphen Functionen*, Vol. 1, pp. 365-370, B.G. Teubner, Leipzig 1897. Reprint: New York, Johnson reprint Corporation (Academic Press 1965).
- [Go] W. Goldman, Invariant functions on Lie groups and Hamiltonian flows of surface group representations, *Invent. Math.* **85** (1986), 263-302.
- [G-M] F. Gonzalez-Acuna, J. M. Montesinos-Amilibia, On the character variety of group representations in $SL(2, \mathbb{C})$ and $PSL(2, \mathbb{C})$, *Math. Z.* **214** (1993), 627-652.

- [Ho] R.D. Horowitz, Characters of free groups represented in two-dimensional special linear group, *Comm. Pure and Appl. Math.* **25** (1972), 635-649.
- [H-P-1] J. Hoste, J.H. Przytycki, A survey of skein modules of 3-manifolds; in Knots 90, *Proceedings of the International Conference on Knot Theory and Related Topics*, Osaka (Japan), August 15-19, 1990, Editor A. Kawauchi, Walter de Gruyter (1992), 363-379.
- [H-P-2] J. Hoste, J.H. Przytycki, The $(2, \infty)$ -skein module of lens spaces; A generalization of the Jones polynomial, *J. Knot Theory Ramifications*, **2**(3) (1993), 321-333.
- [H-P-3] J. Hoste, J.H. Przytycki, The $(2, \infty)$ -skein module of Whitehead manifolds, *J. Knot Theory Ramifications*, **4**(3) (1995), 411-427.
- [Hu] T.W. Hungerford, Algebra, Graduate Texts in Mathematics, Springer-Verlag (1974).
- [L-M] M. Lustig, W. Metzler, Integral representations of $Aut F^n$ and presentation classes of groups, in Combinatorial Methods in Topology and Algebraic Geometry, Ed. J.R. Harper, R. Mandelbaum, Proceedings of a conference in honor of Arthur M. Stone, Rochester 1982, *Contemporary Mathematics*, **44** (1985), 51-67.
- [P-1] J.H. Przytycki, Skein modules of 3-manifolds, *Bull. Polish Acad. Science*, **39** (1-2) (1991), 91-100.
- [P-2] J.H. Przytycki, Knot Theory. A combinatorial approach. Cambridge Univ. Press, to appear.
- [P-S-1] J.H. Przytycki, A.S. Sikora, Skein algebra of a group, to appear in Proceedings of The International Conference in Knot Theory, Warsaw 1995, *Banach Center Publications* **42**
- [P-S-2] J.H. Przytycki, A.S. Sikora, Skein algebras of surfaces, Preprint.
- [Ri] R. Riley, Nonabelian representations of 2-bridge knot groups, *Quart. J. Math Oxford Ser.* **35** (1984), 191-208.
- [S-1] K. Saito, Representation varieties of a finitely generated group into SL_2 or GL_2 , RIMS Publications **958**, Kyoto University (1993).

- [S-2] K. Saito, Character variety of representations of a finitely generated group in SL_2 , Proceedings of the 37th Taniguchi Symposium on Topology and Teichmüller Spaces, Finland 1995. Edited by S. Kojima *et al.*
- [Si] A. S. Sikora, On quaternion representations of groups, in preparation.
- [S-F] A. S. Sikora, C. Frohman, A geometric method in the theory of SL_n -representations of groups, preprint.
- [Th] Le Ty Kuok Tkhang¹², Varieties of Representations And Their Cohomology-Jump Subvarieties for knot groups, *Russian Acad. Sci. Sb. Math.* **78** (1994), No. 1, 187-209.
- [T-1] V. G. Turaev, The Conway and Kauffman modules of the solid torus, *Zap. Nauchn. Sem. Lomi* **167** (1988), 79-89. English translation: J. Soviet Math.
- [Vo] H. Vogt, Sur les invariants, fondamentaux des équations différentielles linéaires du second ordre, *Ann. Sci. Ecole Norm. Sup.* (3) 6, Suppl. 3-72 (1889) (Thèse, Paris).

Authors addresses:

*Department of Mathematics, The George Washington University
2201 G Str. room 428 Funger Hall
Washington, D.C. 20052
email: przytyck@math.gwu.edu*

*Department of Mathematics, University of Maryland
College Park, MD 20742
e-mail: asikora@math.umd.edu*

¹²The proper spelling should be Le Tu Quoc Thang.